Ann. PDE (2018) 4:9
https://doi.org/10.1007/s40818-018-0044-1

MANUSCRIPT

Non-local Functionals Related to the Total Variation and Connections with Image Processing

Haïm Brezis[1,2,3] · **Hoai-Minh Nguyen**[4]

Received: 3 January 2018 / Accepted: 12 January 2018

Abstract We present new results concerning the approximation of the total variation, $\int_\Omega |\nabla u|$, of a function u by non-local, non-convex functionals of the form

$$\Lambda_\delta(u) = \int_\Omega \int_\Omega \frac{\delta\varphi\big(|u(x) - u(y)|/\delta\big)}{|x - y|^{d+1}}\, dx\, dy,$$

as $\delta \to 0$, where Ω is a domain in $\mathbb{R}^d$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function satisfying some appropriate conditions. The mode of convergence is extremely delicate and numerous problems remain open. De Giorgi's concept of Γ-convergence illuminates the situation, but also introduces mysterious novelties. The original motivation of our work comes from Image Processing.

Keywords Total variation · Bounded variation · Non-local functional · Non-convex functional · Γ-Convergence · Sobolev spaces

H. Brezis: Research partially supported by NSF Grant DMS-1207793.

✉ Haïm Brezis
brezis@math.rutgers.edu

Hoai-Minh Nguyen
hoai-minh.nguyen@epfl.ch

1 Department of Mathematics, Hill Center, Busch Campus, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

2 Departments of Mathematics and Computer Science, Technion, Israel Institute of Technology, 32.000 Haifa, Israel

3 Laboratoire Jacques-Louis Lions UPMC, 4 Place Jussieu, 75005 Paris, France

4 EPFL SB MATHAA CAMA, Station 8, 1015 Lausanne, Switzerland

Published online: 24 January 2018

Mathematics Subject Classification 49Q20 · 26B30 · 46E35 · 28A75

Contents

1 Introduction

Throughout this paper, we assume that $\varphi : [0, +\infty) \to [0, +\infty)$ is continuous on $[0, +\infty)$ except at a finite number of points in $(0, +\infty)$ where it admits a limit from the left and from the right. We also assume that $\varphi(0) = 0$ and that $\varphi(t) = \min\{\varphi(t+), \varphi(t-)\}$ for all $t > 0$, so that φ is lower semi-continuous. We assume that the domain $\Omega \subset \mathbb{R}^d$ is either bounded and smooth, or that $\Omega = \mathbb{R}^d$. The case $d = 1$ is already of great interest; many difficulties (and open problems!) occur even when $d = 1$.

Given a measurable function u on Ω, and a small parameter $\delta > 0$, we define the following non-local functionals:

$$\Lambda(u) := \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{d+1}}\, dx\, dy \quad \text{and} \quad \Lambda_\delta(u) := \delta \Lambda(u/\delta). \tag{1.1}$$

Sometimes, it is convenient to be more specific and to write $\Lambda_\delta(u, \varphi, \Omega)$ or $\Lambda_\delta(u, \Omega)$ instead of $\Lambda_\delta(u)$.

Our main goal in this paper is to study the asymptotic behaviour of Λ_δ as $\delta \to 0$. In order to simplify the presentation we make, throughout the paper, the following four basic assumptions on φ:

$$\varphi(t) \le at^2 \text{ in } [0, 1] \text{ for some positive constant } a, \tag{1.2}$$
$$\varphi(t) \le b \text{ in } \mathbb{R}_+ \text{ for some positive constant } b, \tag{1.3}$$
$$\varphi \text{ is non-decreasing,} \tag{1.4}$$

and

$$\gamma_d \int_0^\infty \varphi(t)t^{-2}\,dt = 1, \text{ where } \gamma_d := \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|\,d\sigma \text{ for some } e \in \mathbb{S}^{d-1}. \tag{1.5}$$

A straightforward computation gives

$$\gamma_d = \begin{cases} \dfrac{2}{d-1}|\mathbb{S}^{d-2}| = 2|B^{d-1}| & \text{if } d \ge 3, \\ 4 & \text{if } d = 2, \\ 2 & \text{if } d = 1, \end{cases} \tag{1.6}$$

where $\mathbb{S}^{d-2}$ (resp. B^{d-1}) denotes the unit sphere (resp. ball) in $\mathbb{R}^{d-1}$.

Condition (1.5) is a normalization condition prescribed in order to have (1.9) below with constant 1 in front of $\int_\Omega |\nabla u|$. Denote

$$\mathcal{A} = \big\{\varphi;\ \varphi \text{ satisfies } (1.2) - (1.5)\big\}. \tag{1.7}$$

Note that Λ is **never convex** when $\varphi \in \mathcal{A}$.

We also mention the following additional condition on φ which will be imposed in Sects. 4 and 5:

$$\varphi(t) > 0 \quad \text{for all } t > 0. \tag{1.8}$$

Note that if (1.2) and (1.3) hold, then $\Lambda_\delta(u)$ is finite for every $u \in H^{1/2}(\Omega)$, and in particular for every $u \in C^1(\bar{\Omega})$ when Ω is bounded. Assumptions (1.2) and (1.3) cover a large class of functions φ used in Image Processing (see the list below) and they simplify the presentation of various technical points. In many parts of the paper they can be weaken; in some places it might even be sufficient to assume only that $\int_0^\infty \varphi(t)t^{-2}\,dt < +\infty$. However, assumption (1.4) plays an important role in parts of the proof of Theorem 1. Very little is known without the monotonicity assumption on φ, except when $d = 1$; see Open Problem 1.

Here is a list of specific examples of functions φ that we have in mind. They all satisfy (1.2)-(1.4). In order to achieve (1.5), we choose $\varphi = c_i\tilde{\varphi}_i$ where $\tilde{\varphi}_i$ is taken from the list below and c_i is an appropriate constant.

Example 1

$$\tilde{\varphi}_1(t) = \begin{cases} 0 & \text{if } t \le 1 \\ 1 & \text{if } t > 1. \end{cases}$$

Example 2

$$\tilde{\varphi}_2(t) = \begin{cases} t^2 & \text{if } t \le 1 \\ 1 & \text{if } t > 1. \end{cases}$$

Example 3

$$\tilde{\varphi}_3(t) = 1 - e^{-t^2}.$$

Examples 2 and 3 are motivated by Image Processing (see Sect. 5).

In Sect. 2 we investigate the pointwise limit of Λ_δ as $\delta \to 0$, i.e., the convergence of $\Lambda_\delta(u)$ for fixed u. We first consider the case where $u \in C^1(\bar{\Omega})$, with Ω bounded, and prove (see Proposition 1) that

$$\Lambda_\delta(u) \text{ converges, as } \delta \to 0, \text{ to } TV(u) = \int_\Omega |\nabla u|, \text{ the total variation of } u. \tag{1.9}$$

One may then be tempted to infer that the same conclusion holds for every $u \in W^{1,1}(\Omega)$. Surprisingly, this is not true: for every $d \ge 1$ and for every $\varphi \in \mathcal{A}$, one can construct a function $u \in W^{1,1}(\Omega)$ such that

$$\lim_{\delta \to 0} \Lambda_\delta(u) = +\infty;$$

see Pathology 1 in Sect. 2.2.

If $u \in W^{1,1}(\Omega)$, one may only assert (see Proposition 1) that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) \ge \int_\Omega |\nabla u|,$$

for every $\varphi \in \mathcal{A}$.

When dealing with functions $u \in BV(\Omega)$, the situation becomes even more intricate as explained in Sect. 2.2. In particular, it may happen (see Pathology 3 in Sect. 2.2) that, for some $\varphi \in \mathcal{A}$ and some $u \in BV(\Omega)$,

$$\liminf_{\delta \to 0} \Lambda_\delta(u) < \int_\Omega |\nabla u|.$$

On the other hand, we prove (see (2.25)) that, for every $\varphi \in \mathcal{A}$,

$$\liminf_{\delta \to 0} \Lambda_\delta(u) \ge K \int_\Omega |\nabla u| \quad \forall u \in L^1(\Omega),$$

for some $K \in (0, 1]$ depending only on d and φ, and (see Proposition 2)

$$\limsup_{\delta \to 0} \Lambda_\delta(u) \ge \int_\Omega |\nabla u| \quad \forall u \in L^1(\Omega).$$

Here and throughout the paper, we set $\int_\Omega |\nabla u| = +\infty$ if $u \in L^1(\Omega) \setminus BV(\Omega)$.

All these facts suggest that the mode of convergence of Λ_δ to TV as $\delta \to 0$ is delicate and that pointwise convergence may be deceptive. It turns out that Γ-convergence (in the sense of E. De Giorgi) is the appropriate framework to analyze the asymptotic behavior of Λ_δ as $\delta \to 0$. (For the convenience of the reader, we recall the definition of Γ-convergence in Sect. 3).

Section 3 deals with the following crucial result whose proof is extremely involved.

Theorem 1 *Let $\varphi \in \mathcal{A}$. There exists a constant $K = K(\varphi) \in (0, 1]$, which is independent of Ω such that, as $\delta \to 0$,*

$$\left(\Lambda_\delta\right)\ \Gamma\text{-}converges\ to\ \Lambda_0\ in\ L^1(\Omega), \tag{1.10}$$

where

$$\Lambda_0(u) := K \int_\Omega |\nabla u| \quad \textit{for } u \in L^1(\Omega).$$

Here is a direct consequence of Theorem 1 in the case $d = 1$.

Corollary 1 *Let $u \in L^1(0, 1)$ and $(u_\delta) \subset L^1(0, 1)$ be such that $u_\delta \to u$ in $L^1(0, 1)$. Then*

$$\liminf_{\delta \to 0} \Lambda_\delta(u_\delta, (0, 1)) \geq K(\varphi)|u(t_2) - u(t_1)|, \tag{1.11}$$

for all Lebesgue points $t_1, t_2 \in (0, 1)$ of u.

Despite its simplicity, we do not know an easy proof for Corollary 1 even when $u_\delta \equiv u$, $\varphi = c_1\tilde{\varphi}_1$, and $K(\varphi)$ is replaced by a positive constant independent of u; in this case the result is originally due to J. Bourgain and H.-M. Nguyen [10, Lemma 2].

Remark 1 The constant K may also depend on d (actually we have not investigated whether it really depends on d), but for simplicity we omit this (possible) dependence.

Remark 2 The asymptotic behavior of Λ_δ as $\delta \to 0$ when $\varphi = c_1\tilde{\varphi}_1$ has been extensively studied at the suggestion of H. Brezis; see e.g., [10,42–47,51]. In this particular case, Theorem 1 and Theorem 3 (below) are originally due to H.-M. Nguyen [43], [45], and [46]. The lengthy proof of Theorem 1 borrows numerous ideas from [45], however, the presence of a general function $\varphi \in \mathcal{A}$ in Λ_δ introduces many new challenges, some still unresolved; see, e.g., Open Problems 1 and 2 below.

It would be interesting to remove the monotonicity assumption (1.4) in the definition of $\mathcal{A}$. More precisely, we have

Open Problem 1 *Assume that (1.2), (1.3), and (1.5) hold. Is it true that either the conclusion of Theorem 1 holds, or (Λ_δ) Γ-converges in $L^1(\Omega)$ to 0 as $\delta \to 0$?*

The answer is positive in the one dimensional case [20].

Remark 3 Note that if one removes the monotonicity assumption on φ it may happen that $(\Lambda_\delta) \xrightarrow{\Gamma} 0$ in $L^1(\Omega)$ as $\delta \to 0$. This occurs e.g., when $\operatorname{supp}\varphi \subset\subset (0, +\infty)$. Indeed, given $u \in L^1(\Omega)$, let $(\tilde{u}_\delta)$ be a family of functions converging in $L^1(\Omega)$ to u, as δ goes to 0, such that $\tilde{u}_\delta$ takes its values in the set $m\delta\mathbb{Z}$. Here m is chosen such that $|t| < m/2$ for $t \in \operatorname{supp}\varphi$. It is clear that

$$\Lambda_\delta(\tilde{u}_\delta) = 0, \quad \forall\, \delta > 0.$$

Therefore $\Lambda_\delta \xrightarrow{\Gamma} 0$ in $L^1(\Omega)$.

The appearance of the constant $K = K(\varphi)$ in Theorem 1 is mysterious and somewhat counterintuitive. Assume for example that Ω is bounded and that $u \in C^1(\bar{\Omega})$. We know that $\Lambda_\delta(u) \to \int_\Omega |\nabla u|$ as $\delta \to 0$ (see Proposition 1). On the other hand, it follows from Theorem 1 that there exists a family (u_δ) in $L^1(\Omega)$ such that $u_\delta \to u$ in $L^1(\Omega)$ and $\Lambda_\delta(u_\delta) \to K \int_\Omega |\nabla u|$ as $\delta \to 0$. The reader may wonder how K is determined. This is rather easy to explain, e.g., when $d = 1$ and $\Omega = (0, 1)$; $K(\varphi)$ is given by

$$K(\varphi) = \inf \liminf_{\delta \to 0} \Lambda_\delta(v_\delta), \tag{1.12}$$

where the infimum is taken over all families of functions $(v_\delta)_{\delta \in (0,1)} \subset L^1(0, 1)$ such that $v_\delta \to v_0$ in $L^1(0, 1)$ as $\delta \to 0$ with $v_0(x) = x$ in $(0, 1)$. Unfortunately, formula (1.12) provides very little information about the constant $K(\varphi)$. Taking $v_\delta = v_0$ for all $\delta > 0$, we obtain $K(\varphi) \le 1$ for all φ. Indeed, an easy computation using the normalization (1.5) shows (see Proposition 1) that $\lim_{\delta \to 0} \Lambda_\delta(v_0) = \int_0^1 |v_0'| = 1$. A more sophisticated choice of (v_δ) in [43] yields $K(\varphi) < 1$ when $\varphi = c_1\tilde{\varphi}_1$, for every $d \ge 1$. For the convenience of the reader, we include the proof of this fact in Sect. 3.6. On the other hand, it is nontrivial that $K(\varphi) > 0$ for all $\varphi \in \mathcal{A}$ and $d \ge 1$. It is even less trivial that $\inf_{\varphi \in \mathcal{A}} K(\varphi) > 0$ (see Sect. 3.5).

Here is a challenging question, which is open even when $d = 1$.

Open Problem 2 *Is it always true that $K(\varphi) < 1$ in Theorem 1? Or even better: Is it true that $\sup_{\varphi \in \mathcal{A}} K(\varphi) < 1$?*

We believe that indeed $K(\varphi) < 1$ for every φ. (However, if it turns out that $K(\varphi) = 1$ for some φ's, it would be interesting to characterize such φ's.)

In Sect. 4, we establish the following two compactness results. The first one deals with the level sets of Λ_δ for a **fixed** δ, e.g., for $\delta = 1$.

Theorem 2 *Let $\varphi \in \mathcal{A}$ satisfy (1.8), and let (u_n) be a bounded sequence in $L^1(\Omega)$ such that*

$$\sup_n \Lambda(u_n) < +\infty. \tag{1.13}$$

There exists a subsequence (u_{n_k}) of (u_n) and $u \in L^1(\Omega)$ such that (u_{n_k}) converges to u in $L^1(\Omega)$ if Ω is bounded, resp. in $L^1_{loc}(\mathbb{R}^d)$ if $\Omega = \mathbb{R}^d$.

The second result concerns a sequence $\left(\Lambda_{\delta_n}\right)$ with $\delta_n \to 0$; here (1.8) is not required.

Theorem 3 *Let $\varphi \in \mathcal{A}$, $(\delta_n) \to 0$, and let (u_n) be a bounded sequence in $L^1(\Omega)$ such that*

$$\sup_n \Lambda_{\delta_n}(u_n) < +\infty. \tag{1.14}$$

There exists a subsequence (u_{n_k}) *of* (u_n) *and* $u \in L^1(\Omega)$ *such that* (u_{n_k}) *converges to* u *in* $L^1(\Omega)$ *if* Ω *is bounded, resp. in* $L^1_{loc}(\mathbb{R}^d)$ *if* $\Omega = \mathbb{R}^d$.

In Sect. 5, we consider problems of the form

$$\inf_{u \in L^q(\Omega)} E_\delta(u), \tag{1.15}$$

in the case Ω bounded, where

$$E_\delta(u) = \lambda \int_\Omega |u - f|^q + \Lambda_\delta(u), \tag{1.16}$$

$q \geq 1$, $f \in L^q(\Omega)$ is given, and λ is a fixed positive constant. Our goal is twofold: investigate the existence of minimizers for E_δ (δ being fixed) and analyze their behavior as $\delta \to 0$. The existence of a minimizer in (1.15) is not straightforward since Λ_δ is **not convex** and one cannot invoke the standard tools of Functional Analysis. Theorem 2 implies the existence of a minimizer in (1.15). Next we study the behavior of these minimizers as $\delta \to 0$. More precisely, we prove

Theorem 4 *Assume that* Ω *is bounded, and that* $\varphi \in \mathcal{A}$ *satisfies* (1.8). *Let* $q \geq 1$, $f \in L^q(\Omega)$, *and let* u_δ *be a minimizer of* (1.16). *Then* $u_\delta \to u_0$ *in* $L^q(\Omega)$ *as* $\delta \to 0$, *where* u_0 *is the unique minimizer of the functional* E_0 *defined on* $L^q(\Omega) \cap BV(\Omega)$ *by*

$$E_0(u) := \lambda \int_\Omega |u - f|^q + K \int_\Omega |\nabla u|,$$

and $0 < K \leq 1$ *is the constant coming from Theorem 1.*

Basic ingredients in the proof are the Γ-convergence result (Theorem 1) and the compactness result (Theorem 3).

As explained in Sect. 5, E_δ and E_0 are closely related to functionals used in Image Processing for the purpose of denoising the image f. In fact, E_0 corresponds to the celebrated ROF filter originally introduced by L. I. Rudin, S. Osher and E. Fatemi in [52]. While E_δ (with φ as in Examples 2–3) is reminiscent of filters introduced by L. S. Lee [39] and L. P. Yaroslavsky (see [55,56]). More details can be found in the expository paper by A. Buades, B. Coll, and J. M. Morel [21]; see also [22,23,48,53] where various terms, such as "neighbourhood filters", "non-local means" and "bilateral filters", are used. Some of these filters admit a variational formulation, as explained by S. Kindermann, S. Osher and P. W. Jones in [38]. Theorem 4 says that such filters "converge" to the ROF filter, as $\delta \to 0$, a fact which seems to be new to the experts in Image Processing.

In recent years there has been much interest in the convergence of **convex** non-local functionals to the total variation, going back to the work of J. Bourgain, H. Brezis and P. Mironescu [7] (see Remark 4 below). Related works may be found in [8,12,15,17,18,24,25,31,32,34,40,49,54]. For the convergence of non-local functionals to the perimeter, we mention in particular [4,26], the two surveys [24, Section 5], [34, Section 5.6], and the references therein. As one can see, there is a "family resemblance" with questions studied in our paper. We warn the reader that the non-convexity of Λ_δ is a source of major difficulties. Moreover, new and surprising phenomena emerged over the past fifteen years, in particular the discovery in [43,45] that the Γ-limit and the pointwise limit of (Λ_δ) do not coincide; we refer to [14] for some historical comments. We also mention that a different type of approximation of the BV-norm of a function u, especially suited when u is the characteristic function of a set A, so that its BV-norm is the perimeter of A, has been recently developed in [2] and [3] (with roots in [9]).

Part of the results in this paper are announced in [14,19,47].

After our work was completed and posted on arXiv, we received an interesting paper by C. Antonucci, M. Gobbino, M. Migliorini and N. Picenni [5] concerning the asymptotic behavior of Λ_δ as $\delta \to 0$ specifically when $\varphi = c_1\tilde{\varphi}_1$ and $\Omega = \mathbb{R}^d$. In particular, they obtain the explicit value of $K(c_1\tilde{\varphi}_1) = \ln 2 \approx 0.7$ for every $d \geq 1$. This confirms the conjecture made by H.-M. Nguyen [43] for $d = 1$. They also answer positively Open Problem 3 in Sect. 3.1 (below) when $\varphi = c_1\tilde{\varphi}_1$ and $\Omega = \mathbb{R}^d$. In addition they present a totally new proof of Theorem 1 when $\varphi = c_1\tilde{\varphi}_1$ and $\Omega = \mathbb{R}^d$. It would be desirable to extend their approach to a general function $\varphi \in \mathcal{A}$.

2 Pointwise Convergence of Λ_δ as $\delta \to 0$

2.1 Some Positive Results

The first result in this section is

Proposition 1 *Assume that $\varphi \in \mathcal{A}$. Then*

$$\lim_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u|, \tag{2.1}$$

for all $u \in C^1(\overline{\Omega})$ if Ω is bounded, resp. for all $u \in C^1_c(\mathbb{R}^d)$ if $\Omega = \mathbb{R}^d$. However, if $u \in W^{1,1}(\Omega)$ (with Ω bounded or $\Omega = \mathbb{R}^d$), we can only assert that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) \geq \int_\Omega |\nabla u|, \tag{2.2}$$

and strict inequality may happen (see Pathology 1 in Sect. 2.2).

Remark 4 The convergence of a special sequence of **convex** non-local functionals to the total variation was originally analyzed by J. Bourgain, H. Brezis and P. Mironescu [7] and further investigated in [8,12,15,17,18,31,40,49,54]. More precisely, it has been shown that, for every $u \in L^1(\Omega)$,

$$\lim_{\varepsilon \to 0} J_\varepsilon(u) = \gamma_d \int_\Omega |\nabla u|, \tag{2.3}$$

where

$$J_\varepsilon(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|} \rho_\varepsilon(|x - y|)\, dx\, dy. \tag{2.4}$$

Here γ_d is defined in (1.5), ρ_ε is an arbitrary sequence of radial mollifiers (normalized by the condition $\int_0^\infty \rho_\varepsilon(r) r^{d-1}\, dr = 1$). As the reader can see, (2.1) and (2.3) look somewhat similar. However, the asymptotic analysis of Λ_δ is much more delicate because two basic properties satisfied by J_ε are **not** fulfilled by Λ_δ:

(i) there is **no** constant C such that, e.g., with Ω bounded,

$$\Lambda_\delta(u) \le C \int_\Omega |\nabla u| \quad \forall\, u \in C^1(\bar{\Omega}),\ \forall\, \delta > 0, \tag{2.5}$$

despite the fact $\lim_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u|$ for all $u \in C^1(\bar{\Omega})$. Indeed, if (2.5) held, we would deduce by density the same estimate for every $u \in W^{1,1}(\Omega)$ and this contradicts Pathology 1 in Sect. 2.2.

(ii) $\Lambda_\delta(u)$ is **not** a convex functional.

It is known (see [49]) that the Γ-limit and the pointwise limit of (J_ε) coincide and are equal to $\gamma_d \int_\Omega |\nabla \cdot|$. By contrast, this is not true for Λ_δ since the constant K in Theorem 1 might be less than 1 (e.g., when $\varphi = c_1 \tilde{\varphi}_1$).

Here and in what follows in this paper, given a function $\varphi : [0, +\infty) \to \mathbb{R}$ and $\delta > 0$, we denote φ_δ the function

$$\varphi_\delta(t) = \delta \varphi(t/\delta) \text{ for } t \ge 0.$$

With this notation, one has

$$\Lambda_\delta(u, \varphi) = \Lambda(u, \varphi_\delta).$$

Proof of Proposition 1 We first consider the case $\Omega = \mathbb{R}^d$ and $u \in C^1_{\mathrm{c}}(\mathbb{R}^d)$. Fix $M > 1$ such that $u(x) = 0$ if $|x| \ge M - 1$. We have

$$\begin{aligned}\Lambda_\delta(u) = & \int_{|x|>M} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x-y|^{d+1}}\, dy \\ & + \int_{|x|\le M} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x-y|^{d+1}}\, dy.\end{aligned}$$

Since φ is bounded and

$$\int_{|x|>M} dx \int_{|y|<M-1} \frac{1}{|x-y|^{d+1}}\, dy < +\infty,$$

it follows from the choice of M that

$$\lim_{\delta \to 0} \int_{|x|>M} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x-y|^{d+1}}\, dy = 0. \tag{2.6}$$

Replacing y by $x + z$ and using polar coordinates in the z variable, we find

$$\int_{|x|\le M} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x)-u(y)|)}{|x-y|^{d+1}}\, dy = \int_{|x|\le M} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi_\delta(|u(x+h\sigma)-u(x)|)}{h^2}\, d\sigma. \tag{2.7}$$

We have

$$\int_{|x|\le M} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi_\delta(|u(x+h\sigma)-u(x)|)}{h^2}\, d\sigma = \int_{|x|\le M} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\delta\varphi\Big(|u(x+h\sigma)-u(x)|/\delta\Big)}{h^2}\, d\sigma. \tag{2.8}$$

Rescaling the variable h gives

$$\int_{|x|\le M} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\delta\varphi\Big(|u(x+h\sigma)-u(x)|/\delta\Big)}{h^2}\, d\sigma = \int_{|x|\le M} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi\Big(|u(x+\delta h\sigma)-u(x)|/\delta\Big)}{h^2}\, d\sigma. \tag{2.9}$$

Combining (2.7), (2.8), and (2.9) yields

$$\int_{|x|\le M} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x)-u(y)|)}{|x-y|^{d+1}}\, dy = \int_{|x|\le M} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi\Big(|u(x+\delta h\sigma)-u(x)|/\delta\Big)}{h^2}\, d\sigma. \tag{2.10}$$

Note that

$$\lim_{\delta\to 0} \frac{|u(x+\delta h\sigma)-u(x)|}{\delta} = |\nabla u(x)\cdot\sigma| h \text{ for } (x,\ h,\ \sigma) \in \mathbb{R}^d \times [0,+\infty) \times \mathbb{S}^{d-1}. \tag{2.11}$$

Since φ is continuous at 0 and on $(0,+\infty)$ except at a finite number of points, it follows that

$$\lim_{\delta\to 0} \frac{1}{h^2}\varphi\Big(|u(x+\delta h\sigma)-u(x)|/\delta\Big) = \frac{1}{h^2}\varphi\Big(|\nabla u(x)\cdot\sigma| h\Big) \quad \text{for a.e. } (x,\ h,\ \sigma) \in \mathbb{R}^d \times (0,+\infty) \times \mathbb{S}^{d-1} \tag{2.12}$$

(if $|\nabla u(x) \cdot \sigma|h$ is a point of discontinuity of φ, we may change a little bit h). Rescaling once more the variable h gives

$$\int_0^\infty dh \int_{\mathbb{S}^{d-1}} \frac{1}{h^2} \varphi\Big(|\nabla u(x) \cdot \sigma| h\Big)\, d\sigma = |\nabla u(x)| \int_0^\infty \varphi(t) t^{-2}\, dt \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|\, d\sigma; \tag{2.13}$$

here we have also used the obvious fact that, for every $V \in \mathbb{R}^d$, and for any fixed $e \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} |V \cdot \sigma|\, d\sigma = |V| \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|\, d\sigma. \tag{2.14}$$

Thus, by the normalization condition (1.5), we obtain

$$\int_{|x| \le M} dx \int_0^\infty dh \int_{\mathbb{S}^{d-1}} \frac{1}{h^2} \varphi\Big(|\nabla u(x) \cdot \sigma| h\Big)\, d\sigma = \int_{|x| \le M} |\nabla u|\, dx. \tag{2.15}$$

Define $\hat{\varphi} : [0, \infty) \to \mathbb{R}$ as follows

$$\hat{\varphi}(t) = \begin{cases} (a+b)t^2 & \text{if } 0 \le t \le 1, \\ a+b & \text{if } t > 1, \end{cases}$$

where a and b are the constants in (1.2) and (1.3). Then

$$\varphi \le \hat{\varphi} \text{ on } [0, +\infty). \tag{2.16}$$

We note that

$$\int_0^\infty \hat{\varphi}(t) t^{-2}\, dt < +\infty. \tag{2.17}$$

Since $u \in C^1_{\mathrm{c}}(\mathbb{R}^d)$, it is clear that

$$\frac{|u(x + \delta h \sigma) - u(x)|}{\delta} \le Ch \text{ for } (x, h, \sigma) \in \mathbb{R}^d \times [0, +\infty) \times \mathbb{S}^{d-1}, \tag{2.18}$$

for some positive constant C. On the other hand, by (2.17),

$$\int_{|x| \le M} dx \int_0^\infty dh \int_{\mathbb{S}^{d-1}} \frac{1}{h^2} \hat{\varphi}(Ch)\, d\sigma < +\infty. \tag{2.19}$$

Applying the dominated convergence theorem, and using (2.10), (2.12), (2.15), (2.16), (2.18) and (2.19), we find

$$\lim_{\delta \to 0} \int_{|x| \le M} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{d+1}}\, dy = \int_{|x| \le M} |\nabla u|\, dx. \tag{2.20}$$

Assertion (2.1) now follows from (2.6) and (2.20).

The proof of (2.2) is almost identical, even simpler. In fact (2.2) is an immediate consequence of (2.12) and (2.13), and Fatou's lemma.

We next consider the case where Ω is bounded. Let $D \subset\subset \Omega$ and fix $t > 0$ small enough such that $B(x, t) = \{y \in \mathbb{R}^d;\ |y - x| < t\} \subset\subset \Omega$ for every $x \in D$. We have, for every $u \in W^{1,1}(\Omega)$,

$$\begin{aligned}\Lambda_\delta(u) &\geq \int_D dx \int_{B(x,t)} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{d+1}}\, dy \\ &= \int_D dx \int_0^{t/\delta} \int_{\mathbb{S}^{d-1}} \frac{\varphi\Big(|u(x + \delta h\sigma) - u(x)|/\delta\Big)}{h^2}\, d\sigma\, dh.\end{aligned}$$

By the same method as above, we deduce that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) \geq \int_D |\nabla u| \quad \forall\, u \in W^{1,1}(\Omega); \tag{2.21}$$

which implies (2.2) since $D \subset \Omega$ is arbitrary.

In order to prove (2.1) for every $u \in C^1(\bar{\Omega})$, we write

$$\Lambda_\delta(u) = A_\delta + B_\delta + C_\delta,$$

where

$$\begin{aligned}A_\delta &= \int_D dx \int_{B(x,t)} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{d+1}}\, dy, \\ B_\delta &= \int_D dx \int_{\Omega \setminus B(x,t)} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{d+1}}\, dy,\end{aligned}$$

and

$$C_\delta = \int_{\Omega \setminus D} dx \int_\Omega \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{d+1}}\, dy.$$

By the same method as above, we find

$$\lim_{\delta \to 0} A_\delta = \int_D |\nabla u|. \tag{2.22}$$

On the other hand, we have

$$B_\delta \leq \delta b |\Omega|^2 / t^{d+1}, \tag{2.23}$$

and, as above,

$$C_\delta \leq \delta \int_{\Omega \setminus D} dx \int_\Omega \frac{\hat{\varphi}(L|x - y|/\delta)}{|x - y|^{d+1}}\, dy,$$

where L is the Lipschitz constant of u on Ω. An immediate computation gives

$$C_\delta \leq C |\Omega \setminus D|, \tag{2.24}$$

where C depends only on L, a, b, and d. It is clear that

$$\left|\Lambda_\delta(u) - \int_\Omega |\nabla u|\right| \le \left|A_\delta - \int_D |\nabla u|\right| + B_\delta + C_\delta + \int_{\Omega\setminus D} |\nabla u|.$$

Using (2.22), (2.23), and (2.24), we conclude that

$$\limsup_{\delta\to 0} |\Lambda_\delta(u) - \int_\Omega |\nabla u|| \le C|\Omega\setminus D|;$$

which implies (2.1) since D is arbitrary. The proof is complete. □

Remark 5 We call the attention of the reader that the monotonicity assumption (1.4) on φ has **not** been used in the proof of Proposition 1.

Remark 6 The condition $u \in C^1(\overline{\Omega})$ if Ω is bounded (resp. $u \in C^1_c(\mathbb{R}^d)$ if $\Omega = \mathbb{R}^d$) in (2.1) is much too strong. In fact, the same conclusion holds under the assumption that Ω is bounded and u is Lipschitz (with an identical proof). More generally, equality (2.1) holds e.g., when $u \in W^{1,p}(\Omega)$ for some $p > 1$, and Ω is bounded (see Proposition C1 in Appendix C). It would be interesting to characterize the set

$$\left\{u \in W^{1,1}(\Omega);\ \lim_{\delta\to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u|\right\}.$$

So far we have been dealing with the pointwise convergence of $\Lambda_\delta(u)$ when $u \in W^{1,1}(\Omega)$, but it is natural to ask similar questions when $u \in BV(\Omega)$. As a consequence of Theorem 1, we know that for every $\varphi \in \mathcal{A}$, there exists a constant $K = K(\varphi) \in (0, 1]$ such that

$$\liminf_{\delta\to 0} \Lambda_\delta(u) \ge K \int_\Omega |\nabla u| \quad \forall u \in L^1(\Omega). \tag{2.25}$$

On the other hand, we also have

Proposition 2 *Assume that $\varphi \in \mathcal{A}$. Then*

$$\limsup_{\delta\to 0} \Lambda_\delta(u) \ge \int_\Omega |\nabla u| \quad \forall u \in L^1(\Omega). \tag{2.26}$$

Proof of Proposition 2 It suffices to consider the case

$$F := \limsup_{\delta\to 0} \Lambda_\delta(u) < +\infty. \tag{2.27}$$

We first assume that $u \in L^\infty(\Omega)$. Set

$$A = 2\|u\|_{L^\infty}. \tag{2.28}$$

Fix $0 < \delta_0 < 1$. Set, for $0 < \varepsilon < 1/2$,

$$T(\varepsilon, \delta_0) := \int_0^{\delta_0} \varepsilon \delta^{\varepsilon - 1} \Lambda_\delta(u)\, d\delta = \int_0^{\delta_0} \varepsilon \delta^{\varepsilon - 1}\, d\delta \int_\Omega \int_\Omega \frac{\delta \varphi(|u(x) - u(y)|/\delta)}{|x - y|^{d+1}}\, dx\, dy. \tag{2.29}$$

We next adapt a device from [42]. Using Fubini's theorem and integrating first with respect to δ, we have

$$T(\varepsilon, \delta_0) = \int_\Omega \int_\Omega \frac{\varepsilon |u(x) - u(y)|^{1+\varepsilon}}{|x - y|^{d+1}}\, dx\, dy \int_{|u(x)-u(y)|/\delta_0}^{\infty} \varphi(t) t^{-2-\varepsilon}\, dt.$$

This implies

$$T(\varepsilon, \delta_0) \ge c(\varepsilon, \delta_0) \quad \mathop{\int_\Omega \int_\Omega}_{|u(x)-u(y)| < \delta_0^2} \frac{\varepsilon |u(x) - u(y)|^{1+\varepsilon}}{|x - y|^{d+1}}\, dx\, dy,$$

where

$$c(\varepsilon, \delta_0) = \int_{\delta_0}^{\infty} \varphi(t) t^{-2-\varepsilon}\, dt.$$

It follows from (2.28) that

$$\begin{aligned} T(\varepsilon, \delta_0) \ge c(\varepsilon, \delta_0) \int_\Omega \int_\Omega \frac{\varepsilon |u(x) - u(y)|^{1+\varepsilon}}{|x - y|^{d+1}}\, dx\, dy - c(\varepsilon, \delta_0) \\ \times \mathop{\int_\Omega \int_\Omega}_{|u(x)-u(y)| \ge \delta_0^2} \frac{\varepsilon A^{1+\varepsilon}}{|x - y|^{d+1}}\, dx\, dy. \end{aligned} \tag{2.30}$$

Let $\tau > 0$ be arbitrary small. First choose δ_0 small enough such that

$$\int_{\delta_0}^{\infty} \varphi(t) t^{-2}\, dt \ge \gamma_d^{-1}(1 - \tau) \tag{2.31}$$

and

$$\Lambda_\delta(u) \le F + \tau \quad \forall\, 0 < \delta < \delta_0. \tag{2.32}$$

We next observe that

$$\mathop{\int_\Omega \int_\Omega}_{|u(x)-u(y)| \ge \alpha} \frac{1}{|x - y|^{d+1}}\, dx\, dy < +\infty \quad \forall \alpha > 0. \tag{2.33}$$

Indeed, fix $t_0 > 0$ such that $\varphi(t_0) > 0$ and note

$$\begin{aligned}\Lambda_\delta(u) &\geq \underset{|u(x)-u(y)|\geq\alpha}{\int_\Omega\int_\Omega} \frac{\varphi_\delta(|u(x)-u(y)|)}{|x-y|^{d+1}}\,dx\,dy \\ &\geq \delta\varphi(\alpha/\delta) \underset{|u(x)-u(y)|\geq\alpha}{\int_\Omega\int_\Omega} \frac{1}{|x-y|^{d+1}}\,dx\,dy. \end{aligned} \tag{2.34}$$

Choosing $0 < \delta < \min\{\delta_0, \alpha/t_0\}$ and using (2.32), we obtain (2.33). We deduce from (2.33) that

$$\lim_{\varepsilon\to 0} c(\varepsilon,\delta_0) \underset{|u(x)-u(y)|>\delta_0^2}{\int_\Omega\int_\Omega} \frac{\varepsilon A^{1+\varepsilon}}{|x-y|^{d+1}}\,dx\,dy = 0. \tag{2.35}$$

We next invoke the following lemma which is an immediate consequence of the BBM formula (2.3) applied with $\rho_\varepsilon(t) = \varepsilon t^{\varepsilon-d}\mathbb{1}_{(0,1)}$ (see [17, Proposition 1]). □

Lemma 1 *We have*

$$\liminf_{\varepsilon\to 0} \gamma_d^{-1} \int_\Omega\int_\Omega \frac{\varepsilon|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{d+1}}\,dx\,dy \geq \int_\Omega |\nabla u| \quad \forall\, u \in L^1(\Omega). \tag{2.36}$$

Combining (2.30), (2.31), (2.35), and (2.36) yields

$$\liminf_{\varepsilon\to 0} T(\varepsilon,\delta_0) \geq (1-\tau)\int_\Omega |\nabla u|. \tag{2.37}$$

On the other hand, using (2.29) and (2.32), we find

$$T(\varepsilon,\delta_0) \leq \int_0^{\delta_0} \varepsilon\delta^{\varepsilon-1}(F+\tau)\,d\delta = (F+\tau)\delta_0^\varepsilon,$$

so that

$$\limsup_{\varepsilon\to 0} T(\varepsilon,\delta_0) \leq F+\tau. \tag{2.38}$$

From (2.37) and (2.38), we deduce that

$$F+\tau \geq (1-\tau)\int_\Omega |\nabla u|.$$

Since $\tau > 0$ is arbitrary, we obtain

$$\limsup_{\delta\to 0} \Lambda_\delta(u) \geq \int_\Omega |\nabla u|.$$

The proof is complete in the case $u \in L^\infty(\Omega)$. In the general case, we proceed as follows. Set, for $A > 0$,

$$T_A(s) = \begin{cases} s & \text{if } |s| \le A, \\ A & \text{if } s > A, \\ -A & \text{if } s < -A, \end{cases} \tag{2.39}$$

and

$$u_A = T_A(u).$$

Since φ is non decreasing,

$$\Lambda_\delta(u_A) \le \Lambda_\delta(u).$$

It follows that

$$\int_\Omega |\nabla u_A| \le \limsup_{\delta \to 0} \Lambda_\delta(u_A) \le \limsup_{\delta \to 0} \Lambda_\delta(u).$$

By letting $A \to +\infty$, we obtain

$$\int_\Omega |\nabla u| \le \limsup_{\delta \to 0} \Lambda_\delta(u).$$

The proof is complete. □

2.2 Some Pathologies

Our first example is related to Proposition 1 and shows that inequality (2.2) can be strict. Such a "pathology" was originally discovered by A. Ponce [50] for $\varphi = c_1 \tilde{\varphi}_1$ and presented in [42]. We describe below a simpler function $u \in W^{1,1}(\Omega)$ which is even more pathological.

Pathology 1 *Let $d \ge 1$. There exists $u \in W^{1,1}(\Omega)$ such that*

$$\lim_{\delta \to 0} \Lambda_\delta(u) = +\infty \textit{ for all } \varphi \in \mathcal{A};$$

moreover,

$$\Lambda_\delta(u) = +\infty \quad \forall\, \delta > 0 \textit{ for } \varphi = c_2 \tilde{\varphi}_2.$$

Proof For simplicity, we present only the case $d = 1$ and choose $\Omega = (-1/2, 1/2)$. Define, for $\alpha > 0$,

$$u(x) = \begin{cases} 0 & \text{if } -1/2 < x < 0, \\ |\ln x|^{-\alpha} & \text{if } 0 < x < 1/2. \end{cases}$$

Clearly, $u \in W^{1,1}(\Omega)$. We claim that, for $0 < \alpha < 1$,

$$\lim_{\delta \to 0} \Lambda_\delta(u) = +\infty \text{ for } \varphi = c_1 \tilde{\varphi}_1$$

and, for $0 < \alpha < 1/2$,

$$\Lambda_\delta(u) = +\infty \quad \forall\, \delta > 0 \text{ for } \varphi = c_2\tilde{\varphi}_2.$$

It is clear that the conclusion follows from the claim since for all $\varphi \in \mathcal{A}$ there exist $\alpha, \beta > 0$ such that $\varphi(t) \geq \alpha c_1 \tilde{\varphi}_1(\beta t)$ for all $t > 0$.

It remains to prove the claim. For $\varphi = c_1\tilde{\varphi}_1$, we have

$$\Lambda_\delta(u) \geq c_1 \int_{\substack{0 \\ |u(x)|>\delta}}^{1/2} dx \int_{-1/2}^{0} \frac{\delta}{|x-y|^2}\, dy.$$

For δ sufficiently small, let $x_\delta \in (0, 1/2)$ be the unique solution of $|\ln x|^{-\alpha} = \delta$. A straightforward computation yields

$$\Lambda_\delta(u) \geq c_1\delta \int_{x_\delta}^{1/2} \Big(\frac{1}{x} - \frac{1}{x+1/2}\Big)\, dx \sim \delta|\ln x_\delta| = \delta^{1-1/\alpha} \to +\infty \text{ as } \delta \to 0,$$

if $\alpha < 1$. We now consider the case $\varphi = c_2\tilde{\varphi}_2$. We have, since $|u| \leq 1$,

$$\begin{aligned}\Lambda_\delta(u) &\geq C_\delta \int_0^{1/2} dx \int_{-1/2}^{0} \frac{|u(x)|^2}{|x-y|^2}\, dy \\ &= C_\delta \int_0^{1/2} |\ln x|^{-2\alpha}\Big(\frac{1}{x} - \frac{1}{x+1/2}\Big)\, dx = +\infty,\end{aligned}$$

if $2\alpha < 1$. □

Next, we mention an example of $\varphi \in \mathcal{A}$ and $u \in W^{1,1}$ such that $\lim_{\delta\to 0}\Lambda_\delta(u)$ does **not** exist and the gap between $\liminf_{\delta\to 0}\Lambda_\delta(u)$ and $\limsup_{\delta\to 0}\Lambda_\delta(u)$ is "maximal".

Pathology 2 *Let* $\Omega = (0, 1)$. *There exists a function* $\varphi \in \mathcal{A}$ *and a function* $u \in W^{1,1}(\Omega)$ *such that*

$$\liminf_{\delta\to 0}\Lambda_\delta(u) = \int_\Omega |\nabla u| \quad \textit{and} \quad \limsup_{\delta\to 0}\Lambda_\delta(u) = +\infty. \tag{2.40}$$

The construction is presented in Appendix A. Our next example shows that assertion (2.2) in Proposition 1 may fail for $u \in BV(\Omega)\backslash W^{1,1}(\Omega)$.

Pathology 3 *Let* $\Omega = (0, 1)$. *There exists a continuous function* $\varphi \in \mathcal{A}$ *and a function* $u \in BV(\Omega) \cap C(\bar{\Omega})$ *such that*

$$\liminf_{\delta\to 0}\Lambda_\delta(u) < \int_\Omega |\nabla u|. \tag{2.41}$$

The construction is presented in Appendix B.

Concluding remark: the abundance of pathologies is quite mystifying and a reasonable theory of pointwise convergence of Λ_δ seems out of reach. Fortunately, Γ-convergence saves the situation!

3 Γ-Convergence of Λ_δ as $\delta \to 0$

3.1 Structure of the Proof of Theorem 1

Recall that (see e.g., [11,30]), by definition, a family of functionals $(\Lambda_\delta)_{\delta\in(0,1)}$ defined on $L^1(\Omega)$ (with values in $\mathbb{R}\cup\{+\infty\}$), Γ-converges to Λ_0 in $L^1(\Omega)$ as $\delta \to 0$ if the following two properties hold:

(G1) For every $u \in L^1(\Omega)$ and for every family $(u_\delta)_{\delta\in(0,1)} \subset L^1(\Omega)$ such that $u_\delta \to u$ in $L^1(\Omega)$ as $\delta \to 0$, one has

$$\liminf_{\delta\to 0} \Lambda_\delta(u_\delta) \geq \Lambda_0(u).$$

(G2) For every $u \in L^1(\Omega)$, there exists a family $(\tilde{u}_\delta)_{\delta\in(0,1)} \subset L^1(\Omega)$ such that $\tilde{u}_\delta \to u$ in $L^1(\Omega)$ as $\delta \to 0$, and

$$\limsup_{\delta\to 0} \Lambda_\delta(\tilde{u}_\delta) \leq \Lambda_0(u).$$

Remark 7 It is clear that if $(\delta_n) \subset \mathbb{R}_+$ is any sequence converging to 0 as $n \to +\infty$ and if (Λ_δ) Γ-converges to Λ_0, then (Λ_{δ_n}) also Γ-converges to Λ_0.

The constant K which occurs in Theorem 1 will be defined via a "semi-explicit" construction. More precisely, **fix** any (smooth) function $u \in \mathcal{B} := \Big\{u \in BV(\Omega);\ \int_\Omega |\nabla u| = 1\Big\}$; given any $\varphi \in \mathcal{A} = \{\varphi;\ \varphi \text{ satisfies } (1.2) - (1.5)\}$, set

$$K(u, \varphi, \Omega) = \inf \liminf_{\delta\to 0} \Lambda_\delta(v_\delta), \tag{3.1}$$

where the infimum is taken over all families of functions $(v_\delta)_{\delta\in(0,1)} \subset L^1(\Omega)$ such that $v_\delta \to u$ in $L^1(\Omega)$ as $\delta \to 0$.

We will eventually establish that

$$K(u, \varphi, \Omega) \text{ is independent of } u \text{ and } \Omega;\ \text{ it depends only on } \varphi \text{ and } d, \tag{3.2}$$

and

$$\text{Theorem 1 holds with } K = K(u, \varphi, \Omega). \tag{3.3}$$

A priori, it is very surprising that $K(u, \varphi, \Omega)$ is independent of $u \in \mathcal{B}$. However, a posteriori, if one believes Theorem 1, this becomes natural. Indeed,

$$\liminf_{\delta\to 0} \Lambda_\delta(u_\delta) \geq K \int_\Omega |\nabla u| = K,$$

for every family $(u_\delta) \overset{L^1}{\to} u \in \mathcal{B}$ by $(G1)$, and thus $K(u, \varphi, \Omega) \geq K$. On the other hand, by $(G2)$, there exists a family $(\tilde{u}_\delta) \overset{L^1}{\to} u \in \mathcal{B}$ such that

$$\limsup_{\delta \to 0} \Lambda_\delta(\tilde{u}_\delta) \leq K \int_\Omega |\nabla u| = K,$$

and hence $K(u, \varphi, \Omega) \leq K$.

In view of what we just said, the special choice of u and Ω is irrelevant. For convenience, we define, for $\varphi \in \mathcal{A}$,

$$\kappa(\varphi) = K(U, \varphi, Q), \tag{3.4}$$

where

$$Q = [0, 1]^d \quad \text{and} \quad U(x) := (x_1 + \cdots + x_d)\big/\sqrt{d} \text{ in } Q,$$

so that $\int_Q |\nabla U| = 1$.

Here is a comment about Property (G2). From Property (G2), it follows easily that a stronger form of (G2) holds:

(G2') For every $u \in L^1(\Omega)$, there exists a family $(\hat{u}_\delta)_{\delta \in (0,1)} \subset L^1(\Omega) \cap L^\infty(\Omega)$ such that $\hat{u}_\delta \to u$ in $L^1(\Omega)$ as $\delta \to 0$, and

$$\limsup_{\delta \to 0} \Lambda_\delta(\hat{u}_\delta) \leq \Lambda_0(u).$$

Indeed, it suffices to take

$$\hat{u}_\delta = T_{A_\delta}(\tilde{u}_\delta),$$

where T_A denotes the truncation at the level A (see (2.39)) and $A_\delta \to \infty$. This leads naturally to the following

Open Problem 3 *Given $u \in L^1(\Omega)$, is it possible to find $(\hat{u}_\delta)_{\delta \in (0,1)} \subset L^1(\Omega) \cap C^0(\bar{\Omega})$ (resp. $W^{1,1}(\Omega)$, resp. $L^1(\Omega) \cap C^\infty(\bar{\Omega})$) such that $\hat{u}_\delta \to u$ in $L^1(\Omega)$ as $\delta \to 0$, and*

$$\limsup_{\delta \to 0} \Lambda_\delta(\hat{u}_\delta) \leq \Lambda_0(u)?$$

The question is open even if $\Omega = (0, 1)$, $u(x) = x$, and $\varphi = c_1 \tilde{\varphi}_1$.

The heart of the matter is the non-convexity of φ, so that one **cannot** use convolution. If the answer to Open problem 3 is negative, this would be a kind of Lavrentiev gap phenomenon. In that case, it would be very interesting to study the asymptotics as $\delta \to 0$ of $\Lambda_{\delta|L^1(\Omega) \cap C^0(\bar{\Omega})}$ (with numerous possible variants).

In Sect. 3.2, we prove that

$$0 < \kappa(\varphi) \leq 1 \text{ for all } \varphi \in \mathcal{A}. \tag{3.5}$$

In Sect. 3.3, we prove Property (G2) in Theorem 1.

In Sect. 3.4, we prove Property (G1) in Theorem 1.

In Sect. 3.5, we discuss further properties of $\kappa(\varphi)$. In particular, we show that $\inf_{\varphi \in \mathcal{A}} \kappa(\varphi) > 0$.

In Sect. 3.6, we prove that $\kappa(c_1 \tilde{\varphi}_1) < 1$.

3.2 Proof of (3.5)

By (2.1) in Proposition 1, we have

$$\lim_{\delta \to 0} \Lambda_\delta(U, \varphi, Q) = \int_Q |\nabla U| = 1$$

(the reader may be concerned that Q is not smooth, but the conclusion of Proposition 1 can be easily extended to this case). Hence $\kappa(\varphi) \le 1$ by the definition (3.1) applied with U and Q.

We next claim that $\kappa(\varphi) > 0$. Recall that, by [45, Theorem 2, formulas (1.2) and (1.3)]

$$\liminf_{\delta \to 0} \Lambda_\delta(v_\delta, c_1 \tilde{\varphi}_1, Q) \ge K_1 \int_Q |\nabla U| = K_1,$$

for every sequence $v_\delta \to U$ in $L^1(Q)$ and for some positive constant K_1 (here we also use the fact that convergence in $L^1(Q)$ implies convergence in measure in Q). On the other hand, it is easy to check that for every $\varphi \in \mathcal{A}$ there exist $\alpha, \beta > 0$ such that

$$\varphi(t) \ge \alpha c_1 \tilde{\varphi}_1(\beta t) \quad \forall t > 0.$$

Thus, for every sequence $(v_\delta) \to U$ in $L^1(Q)$,

$$\liminf_{\delta \to 0} \Lambda_\delta(v_\delta, \varphi, Q) \ge \alpha\beta K_1 > 0.$$

Consequently,

$$\kappa(\varphi) > 0.$$

□

3.3 Proof of Property (G2)

The starting point is the definition of $\kappa(\varphi)$ given by (3.1) and (3.4), i.e.,

$$\kappa = \kappa(\varphi) = \inf \liminf_{\delta \to 0} \Lambda_\delta(v_\delta, \varphi, Q),$$

where the infimum is taken over all families of functions $(v_\delta)_{\delta\in(0,1)} \subset L^1(Q)$ such that $v_\delta \to U$ in $L^1(Q)$ as $\delta \to 0$. The goal is to establish (G2) for every domain Ω, i.e., for every $u \in L^1(\Omega)$, there exists a family $(\tilde{u}_\delta)_{\delta\in(0,1)} \subset L^1(\Omega)$ such that $\tilde{u}_\delta \to u$ in $L^1(\Omega)$ as $\delta \to 0$, and

$$\limsup_{\delta\to 0} \Lambda_\delta(\tilde{u}_\delta, \varphi, \Omega) \le \kappa \int_\Omega |\nabla u|.$$

The first step concerns Property (G2) when u is an affine function on a Lipschitz domain Ω; see Lemma 6 for a precise statement. The proof of Lemma 6 is based on a covering lemma (taken from [45, Lemma 3]) and some technical ingredients presented in the first part of Sect. 3.3.1. The next step concerns Property (G2) when the domain is the union of simplices, and u is continuous on the domain, and affine on each simplex. The final step is devoted to the proof of Property (G2) in the general case. Roughly speaking, the idea is to construct a sequence of functions (u_n) and a sequence of domains (Ω_n) such that, for each n, $\Omega \subset \Omega_n$, u_n is continuous on Ω_n and affine on each simplex of Ω_n, Ω_n "tends" to Ω, $u_n \to u$ in $L^1(\Omega)$, and $\|\nabla u_n\|_{L^1(\Omega)} \to \|\nabla u\|_{L^1(\Omega)}$. One concludes by applying the previous step for each n and invoking a diagonal process.

3.3.1 Preliminaries

This section is devoted to several lemmas which are used in the proof of Property (G2) (some of them are also used in the proof of Property (G1)) and are in the spirit of [45, Sections 2 and 3].

In this section, $\varphi \in \mathcal{A}$ is fixed (arbitrary) and $0 < \delta < 1$. We recall that

$$\varphi_\delta(t) = \delta\varphi(t/\delta) \quad \text{for } t \ge 0. \tag{3.6}$$

All subsets A of $\mathbb{R}^d$ are assumed to be measurable and C denotes a positive constant depending only on d unless stated otherwise. For $A \subset \mathbb{R}^d$ and $f : A \mapsto \mathbb{R}$, we denote $\mathrm{Lip}(f, A)$ the Lipschitz constant of f on A.

We begin with

Lemma 2 *Let $A \subset \mathbb{R}^d$ and f, g be measurable functions on A. Define $h_1 = \min(f, g)$ and $h_2 = \max(f, g)$. We have*

$$\Lambda_\delta(h_1, A) \le \Lambda_\delta(f, A) + \iint_{A^2\setminus B_1^2} \frac{\varphi_\delta(|g(x) - g(y)|)}{|x-y|^{d+1}}\, dx\, dy \tag{3.7}$$

and

$$\Lambda_\delta(h_2, A) \le \Lambda_\delta(f, A) + \iint_{A^2\setminus B_2^2} \frac{\varphi_\delta(|g(x) - g(y)|)}{|x-y|^{d+1}}\, dx\, dy, \tag{3.8}$$

where

$$B_1 = \big\{x \in A;\ f(x) \le g(x)\big\} \ \textit{ and } \ B_2 = \big\{x \in A;\ f(x) \ge g(x)\big\}.$$

Assume in addition that g is Lipschitz on A and $L = \mathrm{Lip}(g, A)$. Then

$$\Lambda_\delta(h_1, A) \le \Lambda_\delta(f, A) + CL|A \setminus B_1| \tag{3.9}$$

and

$$\Lambda_\delta(h_2, A) \le \Lambda_\delta(f, A) + CL|A \setminus B_2|. \tag{3.10}$$

Proof It suffices to prove (3.7) and (3.9) since (3.8) and (3.10) are consequences of (3.7) and (3.9) by considering $-f$ and $-g$. We first prove (3.7). One can easily verify that

$$|h_1(x) - h_1(y)| \le \max(|f(x) - f(y)|, |g(x) - g(y)|).$$

This implies (3.7) since $\varphi \ge 0$ and φ is non-decreasing.

To obtain (3.9) from (3.7), one just notes that, since $|g(x) - g(y)| \le L|x - y|$ and φ is non-decreasing,

$$\iint_{A^2 \setminus B_1^2} \frac{\varphi_\delta(|g(x) - g(y)|)}{|x - y|^{d+1}}\, dx\, dy \le \iint_{A^2 \setminus B_1^2} \frac{\varphi_\delta(L|x - y|)}{|x - y|^{d+1}}\, dx\, dy,$$

and, by a change of variables and the normalization condition of φ,

$$\iint_{A^2 \setminus B_1^2} \frac{\varphi_\delta(L|x - y|)}{|x - y|^{d+1}}\, dx\, dy \le 2 \int_{A \setminus B_1} dx \int_{\mathbb{S}^{d-1}} d\sigma \int_0^\infty \frac{\varphi_\delta(Lr)}{r^2}\, dr \le CL|A \setminus B_1|.$$

$\square$

Here is an immediate consequence of Lemma 2.

Corollary 2 *Let $-\infty \le m_1 < m_2 \le +\infty$, $A \subset \mathbb{R}^d$, and f be a measurable function on A. Set*

$$h = \min\big(\max(f, m_1), m_2\big).$$

We have

$$\Lambda_\delta(h, A) \le \Lambda_\delta(f, A). \tag{3.11}$$

Another useful consequence of Lemma 2 is

Corollary 3 *Let $c > 0$, $A \subset \mathbb{R}^d$, and f, g be measurable functions on A. Set $B = \big\{x \in A;\ |f(x) - g(x)| > c\big\}$,*

$$h = \min\big(\max(f, g - c), g + c\big).$$

Assume that g is Lipschitz on A with $L = \mathrm{Lip}(g, A)$. We have

$$\Lambda_\delta(h, A) \le \Lambda_\delta(f, A) + CL|B|.$$

An important consequence of Corollary 3 is

Corollary 4 *Let* $A \subset \mathbb{R}^d$, $g \in L^\infty(A)$, $(\delta_k) \subset \mathbb{R}_+$, *and* $(g_k) \subset L^1(A)$ *be such that* A *is bounded,* g *is Lipschitz, and* $g_k \to g$ *in* $L^1(A)$. *There exists* $(h_k) \subset L^\infty(A)$ *such that* $\|h_k - g\|_{L^\infty(A)} \to 0$ *and*

$$\limsup_{k\to\infty} \Lambda_{\delta_k}(h_k, A) \le \limsup_{k\to\infty} \Lambda_{\delta_k}(g_k, A).$$

Similarly, if $g_\delta \to g$ *in* $L^1(A)$ *as* $\delta \to 0$, *there exists* $(h_\delta) \subset L^\infty(A)$ *such that* $\|h_\delta - g\|_{L^\infty(A)} \to 0$ *and*

$$\limsup_{\delta\to 0} \Lambda_\delta(h_\delta, A) \le \limsup_{\delta\to 0} \Lambda_\delta(g_\delta, A).$$

Proof Set $c_k = \|g_k - g\|_{L^1(A)}^{1/2}$. Then $c_k|A_k| \le \|g_k - g\|_{L^1(A)} = c_k^2$ where $A_k = \{x \in A; |g_k(x) - g(x)| > c_k\}$, so that

$$\lim_{k\to+\infty} c_k = 0 \quad \text{and} \quad \lim_{k\to+\infty} |A_k| = 0. \tag{3.12}$$

Define $h_k = \min(\max(g_k, g - c_k), g + c_k)$ in A. Clearly $\|h_k - g\|_{L^\infty(A)} \le c_k$. Applying Corollary 3, we have

$$\Lambda_{\delta_k}(h_k, A) \le \Lambda_{\delta_k}(g_k, A) + CL|A_k|, \tag{3.13}$$

where L is the Lipschitz constant of g. Letting $k \to +\infty$ in (3.13) and using (3.12), one reaches the conclusion for (h_k). The argument for (g_δ) is the same. □

We now introduce some notations used later. We denote

(i) for $x, y \in \mathbb{R}^d$,

$$|x - y|_\infty = \sup_{i=1,\cdots,d} |x_i - y_i|.$$

(ii) for $c > 0$ and $A \subset \mathbb{R}^d$,

$$A_c = \Big\{x \in A;\ \operatorname{dist}_\infty(x, \partial A) \le c\Big\}, \tag{3.14}$$

where

$$\operatorname{dist}_\infty(x, \partial A) := \inf_{y\in\partial A} |x - y|_\infty.$$

(iii) for $c \in \mathbb{R}$ and for $A, B \subset \Omega$,

$$cA = \{ca\ ;\ a \in A\}$$

and

$$A + B := \{a + b\ ;\ a \in A \text{ and } b \in B\}.$$

We write $A+v$ instead of $A+\{v\}$ for $v \in \mathbb{R}^d$.

We now present an estimate which will be used repeatedly in this paper.

Lemma 3 *Let $c>0$, $g \in L^1(\mathbb{R}^d)$, and let D be a Lipschitz, bounded open subset of $\mathbb{R}^d$. Assume that g is Lipschitz in D_c with $L = \operatorname{Lip}(g, D_c)$. We have*

$$\int_D \int_{\mathbb{R}^d} \frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{d+1}}\,dx\,dy \le \Lambda_\delta(g, D\backslash D_{c/2}) + C_D\Big(Lc + b\delta/c\Big) \text{ for } \delta>0,$$

for some positive constant C_D depending only on D where b is the constant in (1.3).

Proof Set

$$A_1 = (D\backslash D_{3c/4}) \times (D\backslash D_{c/2}), \quad A_2 = D_{3c/4} \times D_c,$$

and

$$A_3 = \Big((D\backslash D_{3c/4}) \times \big(\mathbb{R}^d\backslash(D\backslash D_{c/2})\big)\Big) \cup \Big(D_{3c/4} \times (\mathbb{R}^d\backslash D_c)\Big).$$

It is clear that $D \times \mathbb{R}^d \subset A_1 \cup A_2 \cup A_3$ and $A_1 \subset (D\backslash D_{c/2}) \times (D\backslash D_{c/2})$. A straightforward computation yields

$$\iint_{A_2} \frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{d+1}}\,dx\,dy \le \iint_{A_2} \frac{\varphi_\delta(L|x-y|)}{|x-y|^{d+1}}\,dx\,dy \le C_D Lc.$$

and, since $\varphi \le b$ and if $(x,y) \in A_3$ then $x \in D$ and $|x-y| \ge C_D c$,

$$\begin{aligned}\iint_{A_3} \frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{d+1}}\,dx\,dy &\le \iint_{A_3} \frac{\delta b}{|x-y|^{d+1}}\,dx\,dy \\ &\le \int_D dx \int_{\mathbb{S}^{d-1}} d\sigma \int_{C_D c}^{\infty} \frac{\delta b}{h^2}\,dh \le C_D \delta b/c.\end{aligned}$$

Therefore,

$$\int_D \int_{\mathbb{R}^d} \frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{d+1}}\,dx\,dy \le \Lambda_\delta(g, D\backslash D_{c/2}) + C_D\Big(Lc + b\delta/c\Big).$$

□

We have

Lemma 4 *Let $(\delta_k) \subset \mathbb{R}_+$ and $(g_k) \subset L^1(Q)$ be such that $\delta_k \to 0$ and $g_k \to U$ in $L^1(Q)$. There exist $(c_k) \subset \mathbb{R}_+$ and $(h_k) \subset L^\infty(Q)$ such that*

$$c_k \ge \sqrt{\delta_k}, \quad \lim_{k \to +\infty} c_k = 0,$$
$$\|h_k - U\|_{L^\infty(Q)} \le 2dc_k, \quad \operatorname{Lip}(h_k, Q_{c_k}) \le 1,$$

and

$$\limsup_{k\to+\infty} \Lambda_{\delta_k}(h_k, Q) \le \limsup_{k\to+\infty} \Lambda_{\delta_k}(g_k, Q).$$

Similarly, if $(g_\delta) \subset L^1(Q)$ *is such that* $g_\delta \to U$ *in* $L^1(Q)$*, there exist* $(c_\delta) \subset \mathbb{R}_+$ *and* $(h_\delta) \subset L^\infty(Q)$ *such that*

$$c_\delta \ge \sqrt{\delta_\delta}, \quad \lim_{\delta\to 0} c_\delta = 0,$$
$$\|h_\delta - U\|_{L^\infty(Q)} \le 2dc_\delta, \quad \mathrm{Lip}(h_\delta, Q_{c_\delta}) \le 1,$$

and

$$\limsup_{\delta\to 0} \Lambda_\delta(h_\delta, Q) \le \limsup_{\delta\to 0} \Lambda_\delta(g_\delta, Q).$$

Proof We only give the proof for the sequence (g_k). The proof for the family (g_δ) is the same. By Corollary 4, one may assume that $\|g_k - U\|_{L^\infty(Q)} \to 0$. Set

$$c_k = \max\Big(\|g_k - U\|_{L^\infty(Q)}, \sqrt{\delta_k}\Big), \tag{3.15}$$

denote $g_{0,k} = g_k$, and define

$$\begin{aligned}
g_{1,k}(x) &= \min\Big(\max\Big(g_{0,k}(x), U(0, x_2, \dots, x_d) + 2c_k\Big), U(1, x_2, \dots, x_d) - 2c_k\Big),\\
g_{2,k}(x) &= \min\Big(\max\Big(g_{1,k}(x), U(x_1, 0, \dots, x_d) + 4c_k\Big), U(x_1, 1, \dots, x_d) - 4c_k\Big),\\
&\dots\\
g_{d,k}(x) &= \min\Big(\max\Big(g_{d-1,k}(x), U(x_1, \dots, x_{d-1}, 0) + 2dc_k\Big), U(x_1, \dots, x_{d-1}, 1) - 2dc_k\Big).
\end{aligned} \tag{3.16}$$

From the definition of U, we have

$$\begin{cases} U(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d) + 2ic_k \le U(x) + 2ic_k \\ U(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d) - 2ic_k \ge U(x) - 2ic_k, \end{cases} \quad \text{for } 1 \le i \le d.$$

It follows from (3.16) that, for $1 \le i \le d$,

$$\min\Big(g_{i-1,k}(x), U(x) - 2ic_k\Big) \le g_{i,k}(x) \le \max\Big(g_{i-1,k}(x), U(x) + 2ic_k\Big).$$

Using the fact $U(x) - c_k \le g_{0,k}(x) \le U(x) + c_k$ by (3.15), we obtain, for $1 \le i \le d$,

$$U(x) - 2ic_k \le g_{i,k}(x) \le U(x) + 2ic_k. \tag{3.17}$$

Since $\lim_{k\to+\infty} c_k = 0$, it follows from (3.16) and (3.17) that, for large k,

$$g_{i,k}(x) = \begin{cases} U(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d) + 2ic_k & \text{if } 0 \le x_i \le c_k, \\ U(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d) - 2ic_k, & \text{if } 1 - c_k \le x_i \le 1. \end{cases} \tag{3.18}$$

We derive from (3.16) and (3.18) that $g_{d,k}$ is Lipschitz on Q_{c_k} with a Lipschitz constant 1 ($= |\nabla U|$). We claim that

$$\limsup_{k\to\infty} \Lambda_{\delta_k}(g_{i,k}, Q) \le \limsup_{k\to\infty} \Lambda_{\delta_k}(g_{i-1,k}, Q) \tag{3.19}$$

for all $1 \le i \le d$.

We establish (3.19) for $i = 1$ (the argument is the same for every i). We first apply Lemma 2 with $A = Q$, $f(x) = \max\big(g_{0,k}(x), U(0, x_2, \cdots, x_d) + 2c_k\big)$, and $g(x) = U(1, x_2, \cdots, x_d) - 2c_k$. Recall that

$$Q\backslash B_1 = \big\{x \in Q;\ f(x) > g(x)\big\}.$$

Note that

$$f(x) \le \max\Big(U(x) + c_k, U(0, x_2, \cdots, x_d) + 2c_k\Big) \le U(x) + 2c_k.$$

It follows that if $x \in Q\backslash B_1$ then $U(x) + 2c_k > U(1, x_2, \cdots, x_d) - 2c_k$; this implies $1 - x_1 < 4\sqrt{d}c_k$. Hence $|Q\backslash B_1| \le Cc_k$ and it follows from Lemma 2 that

$$\Lambda_{\delta_k}(g_{1,k}, Q) \le \Lambda_{\delta_k}\Big(\max\big(g_{0,k}(x), U(0, x_2, \cdots, x_d) + 2c_k\big), Q\Big) + Cc_k. \tag{3.20}$$

We next apply Lemma 2 with $A = Q$, $f(x) = g_{0,k}(x)$, $g(x) = U(0, x_2, \cdots, x_d) + 2c_k$, and $B_2 = \{x \in Q;\ f(x) > g(x)\}$. If $x \in Q\backslash B_2$ we have $U(0, x_2, \cdots, x_d) + 2c_k < g_{0,k}$ so that $U(0, x_2, \cdots, x_d) + 2c_k < U(x) - c_k$; this implies $x_1 < 3\sqrt{d}c_k$. Hence $|Q\backslash B_2| \le Cc_k$ and it follows from Lemma 2 that

$$\Lambda_{\delta_k}\Big(\max\big(g_{0,k}(x), U(0, x_2, \cdots, x_d) + 2c_k\big), Q\Big) \le \Lambda_{\delta_k}(g_{0,k}, Q\Big) + Cc_k. \tag{3.21}$$

Claim (3.19) now follows from (3.20) and (3.21) since $\lim_{k\to+\infty} c_k = 0$.

From (3.19), we deduce that

$$\limsup_{k\to\infty} \Lambda_{\delta_k}(g_{d,k}, Q) \le \limsup_{k\to\infty} \Lambda_{\delta_k}(g_k, Q).$$

The conclusion follows by choosing $h_k = g_{d,k}$. □

We next establish the following lemma which plays an important role in the proof of Property (G2).

Lemma 5 *There exist* $(c_\delta) \subset \mathbb{R}_+$ *and* $(g_\delta) \subset L^\infty(Q)$ *such that*

$$c_\delta \ge \sqrt{\delta}, \quad \lim_{\delta\to 0} c_\delta = 0,$$
$$\|g_\delta - U\|_{L^\infty(Q)} \le 2dc_\delta, \quad \mathrm{Lip}(g_\delta, Q_{c_\delta}) \le 1,$$

and

$$\limsup_{\delta \to 0} \Lambda_\delta(g_\delta, Q) \le \kappa.$$

Proof Applying Lemma 4, we derive from the definition of κ that there exist a sequence $(g_k) \subset L^\infty(Q)$ and two sequences (δ_k), $(c_k) \subset \mathbb{R}_+$ such that

$$\lim_{k \to +\infty} \delta_k = \lim_{k \to +\infty} c_k = 0, \quad c_k \ge \sqrt{\delta_k}, \tag{3.22}$$

$$\|g_k - U\|_{L^\infty(Q)} \le 2dc_k, \quad \mathrm{Lip}(g_k, Q_{c_k}) \le 1, \tag{3.23}$$

and

$$\limsup_{k \to +\infty} \Lambda_{\delta_k}(g_k, Q) \le \kappa. \tag{3.24}$$

We next construct a family $(h_\delta) \subset L^\infty(Q)$ such that

$$\|h_\delta - U\|_{L^\infty(Q)} \to 0 \quad \text{and} \quad \limsup_{\delta \to 0} \Lambda_\delta(h_\delta, Q) \le \kappa. \tag{3.25}$$

Let (τ_k) be a strictly decreasing positive sequence such that $\tau_k \le c_k \delta_k$. For each δ small, let k be such that $\tau_{k+1} < \delta \le \tau_k$ and define $m_1 = \delta_k/\delta \ge 1/c_k$ and $m = [m_1]$. As usual, for $a > 0$, $[a]$ denotes the largest integer $\le a$. Define $h_\delta^{(1)} : [0, m]^d \to \mathbb{R}$ as follows

$$h_\delta^{(1)}(y) = \frac{\sum_{i=1}^d [y_i]}{\sqrt{d}} + g_k(x) \text{ with } x = (y_1 - [y_1], \cdots, y_d - [y_d]). \tag{3.26}$$

For $\alpha \in \mathbb{N}^d$ and $c \ge 0$, set

$$Q_{+\alpha} := Q + (\alpha_1, \cdots, \alpha_d), \quad Q_{+\alpha,c} := Q_c + (\alpha_1, \cdots, \alpha_d),$$
$$\text{and} \quad D_{+\alpha,c} := Q_{+\alpha} \setminus Q_{+\alpha,c}.$$

Define

$$Y = \mathbb{N}^d \cap [0, m-1]^d \quad \text{and} \quad B = \bigcup_{\alpha \in Y} \Big(Q_{+\alpha, c_k} \setminus Q_{+\alpha, c_k/2} \Big).$$

We claim that

$$\mathrm{Lip}(h_\delta^{(1)}, B) \le C. \tag{3.27}$$

Indeed, it is clear from (3.23) and (3.26) that

$$\mathrm{Lip}(h_\delta^{(1)}, Q_{+\alpha, c_k} \setminus Q_{+\alpha, c_k/2}) \le 1 \text{ for } \alpha \in Y.$$

On the other hand, if $y \in Q_{+\alpha, c_k} \setminus Q_{+\alpha, c_k/2}$ and $y' \in Q_{+\alpha', c_k} \setminus Q_{+\alpha', c_k/2}$ with $\alpha \neq \alpha'$ then $c_k \le C|y - y'|$ so that

$$|h_\delta^{(1)}(y) - h_\delta^{(1)}(y')| \le |h_\delta^{(1)}(y) - U(y)| + |h_\delta^{(1)}(y') - U(y')| + |U(y) - U(y')|$$
$$\overset{\text{by } (3.23)}{\le} |U(y) - U(y')| + 4dc_k \le |y - y'| + 4dc_k \le C|y - y'|.$$

Claim (3.27) follows.

By classical Lipschitz extension it follows from (3.27) that there exists $h_\delta^{(2)} : \mathbb{R}^d \mapsto \mathbb{R}$ such that $h_\delta^{(2)} = h_\delta^{(1)}$ on B and

$$\operatorname{Lip}(h_\delta^{(2)}, \mathbb{R}^d) \le C. \tag{3.28}$$

Define, for $x \in \mathbb{R}^d$,

$$h_\delta^{(3)}(x) = \begin{cases} h_\delta^{(1)}(x) & \text{if } x \in D_{+\alpha, c_k/2} \text{ for some } \alpha \in Y, \\ h_\delta^{(2)}(x) & \text{otherwise,} \end{cases} \tag{3.29}$$

and set

$$h_\delta(x) = \frac{1}{m_1} h_\delta^{(3)}(mx) \text{ in } [0, 1]^d.$$

Since $\delta = \delta_k / m_1$, by a change of variables, we obtain

$$\Lambda_\delta(h_\delta, Q) = \frac{\delta}{\delta_k} \Lambda_{\delta_k}\big(h_\delta^{(3)}(m\,\cdot), Q\big) = \frac{m^{1-d}}{m_1} \Lambda_{\delta_k}\big(h_\delta^{(3)}, [0, m]^d\big). \tag{3.30}$$

We next estimate $\Lambda_{\delta_k}\big(h_\delta^{(3)}, [0, m]^d\big)$. For $\alpha \in Y$, applying Lemma 3 with $c = c_k$, $D = Q_{+\alpha}$ and $g = h_\delta^{(3)}$, we have

$$\iint\limits_{Q_{+\alpha} \times [0,m]^d} \frac{\varphi_\delta(|h_\delta^{(3)}(x) - h_\delta^{(3)}(y)|)}{|x - y|^{d+1}} \, dx \, dy \le \Lambda_\delta\big(h_\delta^{(3)}, D_{+\alpha, c_k/2}\big) + C(c_k + b\delta/c_k). \tag{3.31}$$

From (3.26) and (3.29), we obtain

$$\Lambda_\delta(h_\delta^{(3)}, D_{+\alpha, c_k/2}) = \Lambda_\delta(g_k, D_{+(0,\cdots,0), c_k/2}) \le \Lambda_\delta(g_k, Q). \tag{3.32}$$

Since

$$\int_{[0,m]^d} \int_{[0,m]^d} \cdots = \sum_{\alpha \in Y} \int_{Q_\alpha} \int_{[0,m]^d} \cdots,$$

it follows from (3.31) and (3.32) that

$$\Lambda_{\delta_k}(h_\delta^{(3)}, [0, m]^d) \le m^d \Lambda_{\delta_k}(g_k, Q) + C m^d (c_k + b\delta_k / c_k). \tag{3.33}$$

Since $m \le m_1$ and $c_k \ge \delta_k^{1/2}$ by (3.22), we deduce from (3.30) and (3.33) that

$$\Lambda_\delta(h_\delta, Q) \le \Lambda_\delta(g_k, Q) + C(c_k + b\delta_k^{\frac{1}{2}}). \tag{3.34}$$

Combining (3.22), (3.24), and (3.34) yields

$$\limsup_{\delta \to 0} \Lambda_\delta(h_\delta, Q) \le \kappa.$$

We next claim that

$$\|h_3^{(\delta)} - U\|_{L^\infty([0,m]^d)} \le Cc_k. \tag{3.35}$$

Indeed, for $y \in [0, m]^d$, we have, by (3.26),

$$|h_1^{(\delta)}(y) - U(y)| = |g_k(x) - U(x)| \text{ where } x = (y_1 - [y_1], \cdots, y_d - [y_d]).$$

It follows from (3.23) that

$$\|h_1^{(\delta)} - U\|_{L^\infty([0,m]^d)} \le Cc_k. \tag{3.36}$$

On the other hand, for $y \in [0, m]^d \setminus \bigcup_{\alpha \in Y} D_{+\alpha, c_k/2}$, let $\hat{y} \in B$ such that $|y - \hat{y}| \le c_k$. Since $h_\delta^{(2)}(\hat{y}) = h_\delta^{(1)}(\hat{y})$, it follows from (3.28) and (3.35) that

$$|h_2^{(\delta)}(y) - U(y)| \le |h_2^{(\delta)}(y) - h_2^{(\delta)}(\hat{y})| + |h_1^{(\delta)}(\hat{y}) - U(\hat{y})| + |U(\hat{y}) - U(y)| \le Cc_k \tag{3.37}$$

Claim (3.35) now follows from (3.36) and (3.37).

Using (3.35), we derive from the definition of h_δ and the facts that $m_1 \ge 1/c_k$ and $c_k \to 0$ that $\|h_\delta - U\|_{L^\infty(Q)} \to 0$. Hence (3.25) is established.

The conclusion now follows from (3.25) and Lemma 4. □

We next establish

Lemma 6 *Let S be an open bounded subset of $\mathbb{R}^d$ with Lipschitz boundary and let g be an affine function defined on S. Then*

$$\inf \liminf_{\delta \to 0} \Lambda_\delta(g_\delta, S) = \kappa |\nabla g||S|, \tag{3.38}$$

where the infimum is taken over all families $(g_\delta)_{\delta \in (0,1)} \subset L^1(S)$ such that $g_\delta \to g$ in $L^1(S)$. Moreover, there exists a family $(h_\delta) \subset L^\infty(S)$ such that $\|h_\delta - g\|_{L^\infty(S)} \to 0$ and

$$\lim_{\delta \to 0} \Lambda_\delta(h_\delta, S) = \kappa |\nabla g||S|.$$

Proof Note that if $T : \mathbb{R}^d \to \mathbb{R}^d$ is an affine conformal transformation, i.e.,

$$T(x) = aRx + b \text{ in } \mathbb{R}^d$$

for some $a > 0$, some linear unitary operator $R : \mathbb{R}^d \to \mathbb{R}^d$, and for some $b \in \mathbb{R}^d$, then, for a measurable subset D of $\mathbb{R}^d$ and $f \in L^1(D)$,

$$\Lambda_\delta(f, D) = a^{1-d}\Lambda_\delta(f \circ T^{-1}, T(D)),$$

by a change of variables.

Using a transformation T as above, we may write $g \circ T^{-1} = U$. Then

$$\Lambda_\delta(g_\delta, S) = a^{1-d}\Lambda_\delta(g_\delta \circ T^{-1}, T(S))$$

and

$$|\nabla g||S| = a^{1-d}|T(S)|$$

Hence, it suffices to prove Lemma 6 for $g = U$.

Denote m the LHS of (3.38). Since $|\nabla g| = |\nabla U| = 1$, (3.38) becomes

$$m = \kappa|S|. \tag{3.39}$$

The proof of (3.39) is based on a covering lemma [45, Lemma 3] (applied first with $\Omega = S$ and $B = Q$ and then with $\Omega = Q$ and $B = S$) which asserts that

(i) There exists a sequence of disjoint sets $(Q_k)_{k\in\mathbb{N}}$ such that Q_k is the image of Q by a dilation and a translation, $Q_k \subset S$ for all k, and

$$|S| = \sum_{k\in\mathbb{N}} |Q_k|.$$

(ii) There exists a sequence of disjoint sets $(S_k)_{k\in\mathbb{N}}$ such that S_k is the image of S by a dilatation and a translation, $S_k \subset Q$ for all k, and

$$|Q| = \sum_{k\in\mathbb{N}} |S_k|.$$

We first claim that

$$m \geq \kappa|S|.$$

Indeed, let (Q_k) be the sequence of disjoint sets in i).

Clearly,

$$\Lambda_\delta(g_\delta, S) \geq \sum_{k\in\mathbb{N}} \Lambda_\delta(g_\delta, Q_k). \tag{3.40}$$

Fix $k \in \mathbb{N}$ and let $a_k > 0$ and $b_k \in \mathbb{R}^d$ be such that $Q_k = a_k Q + b_k$. Then $|Q_k| = a_k^d$ and, by a change of variables,

$$\Lambda_\delta(g_\delta, a_k Q + b_k) = a_k^d \Lambda_{\delta/a_k}(\hat{g}_\delta, Q) \text{ where } \hat{g}_\delta(x) = \frac{1}{a_k} g_\delta(a_k x + b_k). \tag{3.41}$$

From the definition of κ, we have

$$\liminf_{\delta\to 0} \Lambda_{\delta/a_k}(\hat{g}_\delta, Q) \geq \kappa. \tag{3.42}$$

We deduce from (3.41) and (3.42) that

$$\liminf_{\delta\to 0} \Lambda_\delta(g_\delta, Q_k) \geq \kappa|Q_k|. \tag{3.43}$$

Combining (3.40) and (3.43) yields

$$\liminf_{\delta\to 0} \Lambda_\delta(g_\delta, S) \geq \kappa|S|;$$

which implies $m \geq \kappa|S|$. Similarly, using $ii)$ one can show that $\kappa|S| \geq m$. We thus obtain (3.39).

It remains to prove that there exists a family (h_δ) such that $\|h_\delta - g\|_{L^\infty(S)} \to 0$ and

$$\lim_{\delta\to 0} \Lambda_\delta(h_\delta, S) = \kappa|\nabla g||S|. \tag{3.44}$$

As above, we can assume that $g = U$. Let $\hat{Q}$ be the image of Q by a dilatation and a translation such that $S \subset\subset \hat{Q}$. By Lemma 5 and a change of variables, there exists a family (h_δ) such that $h_\delta \to U$ in $L^1(\hat{Q})$ and

$$\lim_{\delta\to 0} \Lambda_\delta(h_\delta, \hat{Q}) = \kappa|\hat{Q}|.$$

On the other hand, we have, by (3.38),

$$\liminf_{\delta\to 0} \Lambda_\delta(h_\delta, \hat{Q}\setminus S) \geq \kappa|\hat{Q}\setminus S|.$$

Moreover,

$$\Lambda_\delta(h_\delta, \hat{Q}) \geq \Lambda_\delta(h_\delta, S) + \Lambda_\delta(h_\delta, \hat{Q}\setminus S).$$

It follows that

$$\limsup_{\delta\to 0} \Lambda_\delta(h_\delta, S) \leq \kappa|S|,$$

which implies (3.44) by (3.38). □

Throughout the rest of Sect. 3.3, we let $A_1, A_2, \ldots, A_m$ be disjoint open $(d+1)$-simplices in $\mathbb{R}^d$ such that every coordinate component of any vertex of A_i is equal to 0 or 1,

$$\bar{Q} = \bigcup_{i=1}^{m} \bar{A}_i,$$

and

$$A_1 = \Big\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d;\ x_i > 0 \text{ for all } 1 \le i \le d, \text{ and } \sum_{i=1}^{d} x_i < 1 \Big\}.$$

We also denote $A_{\ell,c}$ the set $(A_\ell)_c$ (see (3.14)).

The following lemma is a variant of Lemma 5 for $\{A_\ell\}_{\ell=1}^m$.

Lemma 7 *Let $\ell \in \{1, \dots, m\}$ and g be an affine function defined on A_ℓ such that its normal derivative $\frac{\partial g}{\partial n} \neq 0$ along the boundary of A_ℓ, where n denotes the inward normal. There exist a family $(g_\delta) \subset L^\infty(A_\ell)$ and a family $(c_\delta) \subset \mathbb{R}_+$ such that*

$$c_\delta \ge \sqrt{\delta}, \quad \lim_{\delta \to 0} c_\delta = 0,$$
$$\|g_\delta - g\|_{L^\infty(A_\ell)} \le C_d |\nabla g| c_\delta, \quad \mathrm{Lip}(g_\delta, A_{\ell, c_\delta}) \le |\nabla g|,$$

and

$$\limsup_{\delta \to 0} \Lambda_\delta(g_\delta, A_\ell) \le \kappa |\nabla g| |A_\ell|.$$

Proof For notational ease, we assume that $\ell = 1$. The proof is in the spirit of the one of Lemma 4. By Lemma 6, there exists a family $(h_\delta) \subset L^\infty(A_1)$ such that $\|h_\delta - g\|_{L^\infty(A_1)} \to 0$ and

$$\lim_{\delta \to 0} \Lambda_\delta(h_\delta, A_1) = \kappa |\nabla g| |A_1|. \tag{3.45}$$

Set

$$c_\delta = \max\big(\|h_\delta - g\|_{L^\infty(A_1)}, \sqrt{\delta}\big) \quad \text{and} \quad l_\delta = 2|\nabla g| c_\delta.$$

Denote $h_{0,\delta} = h_\delta$, and define, for $i = 1, 2, \dots, d$, and $x \in A_1$,

$$h_{i,\delta}(x) = \begin{cases} \max\big(h_{i-1,\delta}(x), g(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d) + i l_\delta\big) \text{ if } \frac{\partial g}{\partial x_i} > 0, \\ \min\big(h_{i-1,\delta}(x), g(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d) - i l_\delta\big) \text{ if } \frac{\partial g}{\partial x_i} < 0. \end{cases} \tag{3.46}$$

Set $e = (\frac{1}{\sqrt{d}}, \dots, \frac{1}{\sqrt{d}})$ and define, for $x \in A_1$,

$$h_{d+1,\delta}(x) = \begin{cases} \max\big(h_{d,\delta}(x), g(z(x)) + (d+1) l_\delta\big) \text{ if } \frac{\partial g}{\partial e} < 0, \\ \min\big(h_{d,\delta}(x), g(z(x)) - (d+1) l_\delta\big) \text{ if } \frac{\partial g}{\partial e} > 0. \end{cases} \tag{3.47}$$

Here for each $x \in A_1$, $z(x) := x - \langle x, e \rangle e + e$ (the projection of x on the hyperplane P which is orthogonal to e and contains e).

As in the proof of Lemma 4, we have the following three assertions, for $x \in A_1$, (i) for $1 \le i \le d+1$,

$$g(x) - i\ell_\delta \le h_{i,\delta}(x) \le g(x) + i\ell_\delta,$$

(ii) for $1 \leq i \leq d$ and $0 \leq x_i \leq c_\delta$,

$$h_{i,\delta}(x) = \begin{cases} g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) + i l_\delta \text{ if } \frac{\partial g}{\partial x_i} > 0, \\ g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) - i l_\delta \text{ if } \frac{\partial g}{\partial x_i} < 0, \end{cases}$$

(iii) for $|x - z(x)| \leq c_\delta$,

$$h_{d+1,\delta}(x) = \begin{cases} g(z(x)) + (d+1) l_\delta \text{ if } \frac{\partial g}{\partial e} < 0, \\ g(z(x)) - (d+1) l_\delta \text{ if } \frac{\partial g}{\partial e} > 0. \end{cases}$$

It follows that $h_{d+1,\delta}$ is Lipschitz on A_{1,c_δ} with Lipschitz constant $|\nabla g|$. As in the proof of Lemma 4, one has, for $0 \leq i \leq d$,

$$\limsup_{\delta \to 0} \Lambda_\delta(h_{i+1,\delta}, A_1) \leq \limsup_{\delta \to 0} \Lambda_\delta(h_{i,\delta}, A_1);$$

which implies, by (3.45),

$$\limsup_{\delta \to 0} \Lambda_\delta(h_{d+1,\delta}, A_1) \leq \limsup_{\delta \to 0} \Lambda_\delta(h_{0,\delta}, A_1) = \limsup_{\delta \to 0} \Lambda_\delta(h_\delta, A_1) = \kappa |\nabla g| |A_1|.$$

The conclusion now holds for $g_\delta = h_{d+1,\delta}$. □

We end this section with the following result which is a consequence of Lemma 7 by a change of variables.

Definition 1 For each $k \in \mathbb{N}$, a set K is called a k-sim of $\mathbb{R}^d$ if there exist $z \in \mathbb{Z}^d$ and $\ell \in \{1, 2, \ldots, m\}$ such that $K = \frac{1}{2^k} A_\ell + \frac{z}{2^k}$.

We have

Corollary 5 *Let K be a k-sim of $\mathbb{R}^d$ and g be an affine function defined on K such that $\frac{\partial g}{\partial n} \neq 0$ along the boundary of K. There exist a family $(g_\delta) \subset L^\infty(K)$ and a family $(c_\delta) \subset \mathbb{R}_+$ such that*

$$c_\delta \geq C_k \sqrt{\delta}, \quad \lim_{\delta \to 0} c_\delta = 0,$$
$$\|g_\delta - g\|_{L^\infty(K)} \leq C_k |\nabla g| c_\delta, \quad \mathrm{Lip}(g_\delta, K_{c_\delta}) \leq C_k |\nabla g|,$$

and

$$\limsup_{\delta \to 0} \Lambda_\delta(g_\delta, K) \leq \kappa |\nabla g| |K|.$$

In Corollary 5 and Sect. 3.3.2 below, C_k denotes a positive constant depending only on k and d and can be different from one place to another.

3.3.2 Proof of Property (G2)

Our goal is to show that (G2) holds with $K = \kappa$, i.e.,

(G2) For every $u \in L^1(\Omega)$, there exists a family $(\tilde{u}_\delta)_{\delta \in (0,1)} \subset L^1(\Omega)$ such that $\tilde{u}_\delta \to u$ in $L^1(\Omega)$ as $\delta \to 0$, and

$$\limsup_{\delta \to 0} \Lambda_\delta(\tilde{u}_\delta) \le \kappa \int_\Omega |\nabla u|.$$

We consider the case $\Omega = \mathbb{R}^d$ and the case where Ω is bounded separately.

Case 1: $\Omega = \mathbb{R}^d$. The proof is divided into two steps. Given $k \in \mathbb{N}$, set

$$R_k := \left\{ u \in C_c^0(\mathbb{R}^d) \,\middle|\, \begin{array}{l} u \text{ is affine on each} k\text{-sim and } \partial u/\partial n \neq 0 \text{ along the} \\ \text{boundary of each } k\text{-sim, unless } u \text{ is constant on that } k \text{ -sim} \end{array} \right\}. \tag{3.48}$$

Step 1. We prove Property (G2) when $u \in R_k$ and $k \in \mathbb{N}$ is arbitrary but fixed. Set

$$\mathcal{K} = \{K \text{ is a } k\text{-sim and } u \text{ is not constant on } K\}.$$

From now on in the proof of Step 1, K denotes a k-sim. By Corollary 5, for each $K \in \mathcal{K}$, there exist $(u_{K,\delta}) \subset L^\infty(K)$ and $(c_{K,\delta}) \subset \mathbb{R}_+$ such that

$$c_{K,\delta} \ge C_k \sqrt{\delta}, \quad \lim_{\delta \to 0} c_{K,\delta} = 0, \tag{3.49}$$

$$\|u_{K,\delta} - u\|_{L^\infty(K)} \le C_k \|\nabla u\|_{L^\infty(\mathbb{R}^d)} c_{K,\delta}, \quad \mathrm{Lip}(u_{K,\delta}, K_{c_{K,\delta}}) \le C_k \|\nabla u\|_{L^\infty(\mathbb{R}^d)}, \tag{3.50}$$

and

$$\limsup_{\delta \to 0} \Lambda_\delta(u_{K,\delta}, K) \le \kappa \int_K |\nabla u|\, dx. \tag{3.51}$$

For each δ, let u_δ be a function defined in $\mathbb{R}^d$ such that

$$u_\delta = u_{K,\delta} \text{ in } K \backslash K_{c_{K,\delta}/2} \text{ for } K \in \mathcal{K}, \quad u_\delta = u \text{ in } K \text{ for } K \notin \mathcal{K}, \tag{3.52}$$

and

$$|\nabla u_\delta(x)| \le C_k \|\nabla u\|_{L^\infty(\mathbb{R}^d)} \quad \text{for } x \in \mathbb{R}^d \backslash \bigcup_{K \in \mathcal{K}} (K \backslash K_{c_{K,\delta}/2}). \tag{3.53}$$

Such a u_δ exists by (3.50) via standard Lipschitz extension. Applying Lemma 3 with $D = K$ and $g = u_\delta$, we have, by (3.50),

$$\begin{aligned} \iint_{K \times \mathbb{R}^d} \frac{\varphi_\delta(|u_\delta(x) - u_\delta(y)|)}{|x - y|^{d+1}}\, dx\, dy &\le \Lambda_\delta\big(u_\delta, K \backslash K_{c_{K,\delta}/2}\big) \\ &\quad + C_k(\|\nabla u\|_{L^\infty(\mathbb{R}^d)} c_{K,\delta} + b\delta / c_{K,\delta}). \end{aligned} \tag{3.54}$$

From the definition of u_δ, there exists $R > 1$, independent of δ, such that $u_\delta = u = 0$ in $\mathbb{R}^d \backslash B_R$. We have, for some $b > 0$ (see (1.3)),

$$\iint\limits_{(\mathbb{R}^d \backslash B_{R+1}) \times \mathbb{R}^d} \frac{\varphi_\delta(|u_\delta(x) - u_\delta(y)|)}{|x-y|^{d+1}}\, dx\, dy \\ \leq \iint\limits_{B_R \times (\mathbb{R}^d \backslash B_{R+1})} \frac{\delta b}{|x-y|^{d+1}}\, dy\, dx \leq C_d R^d \delta b. \tag{3.55}$$

Combining (3.51), (3.52), (3.54), and (3.55) yields

$$\limsup_{\delta \to 0} \Lambda_\delta(u_\delta, \mathbb{R}^d) \leq \kappa \int_{\mathbb{R}^d} |\nabla u|\, dx.$$

We next claim that $u_\delta \to u$ in $L^1(\mathbb{R}^d)$. Indeed, for $x \in K_{c_{K,\delta}/2}$ for some $K \in \mathcal{K}$, let $\hat{x} \in K_{c_{K,\delta}} \backslash K_{c_{K,\delta}/2}$ be such that $|\hat{x} - x| \leq c_{K,\delta}$. We have

$$|u_\delta(x) - u(x)| \leq |u_\delta(x) - u_\delta(\hat{x})| + |u_\delta(\hat{x}) - u(\hat{x})| + |u(\hat{x}) - u(x)| \leq C_k \|\nabla u\|_{L^\infty(\mathbb{R}^d)} c_{K,\delta}.$$

This implies, for $K \in \mathcal{K}$,

$$\lim_{\delta \to 0} \|u_\delta - u\|_{L^\infty(K)} = 0,$$

Since $u_\delta = u$ in K for $K \notin \mathcal{K}$, we deduce that

$$\lim_{\delta \to 0} \|u_\delta - u\|_{L^1(\mathbb{R}^d)} = 0.$$

The proof of Step 1 is complete.

Step 2. We prove Property (G2) for a general $u \in L^1(\mathbb{R}^d)$. Without loss of generality, one may assume that $u \in BV(\mathbb{R}^d)$ since there is nothing to prove otherwise. Let $(u_n) \subset C_c^\infty(\mathbb{R}^d)$ be such that (u_n) converges to u in $L^1(\mathbb{R}^d)$ and $\|\nabla u_n\|_{L^1(\mathbb{R}^d)} \to \int_{\mathbb{R}^d} |\nabla u|$ as $n \to +\infty$. We next use

Lemma 8 *Let $v \in C_c^1(\mathbb{R}^d)$ with* $\operatorname{supp} v \subset B_R$ *for some $R > 0$. There exists a sequence $(v_m) \subset W^{1,\infty}(\mathbb{R}^d)$ such that $v_m \subset R_m$,* $\operatorname{supp} v_m \subset B_R$ *for large m and $v_m \to v$ in $W^{1,1}(\mathbb{R}^d)$ as $m \to +\infty$.*

Proof of Lemma 8 There exist a sequence $(k_m) \subset \mathbb{N}$ and a sequence $(v_m) \subset W^{1,\infty}(\mathbb{R}^d)$ with $\operatorname{supp} v_m \subset B_R$ for large m such that

(i) $v_m \to v$ in $W^{1,1}(\mathbb{R}^d)$ as $m \to +\infty$.
(ii) v_m is affine on each m-sim.

This fact is standard in finite element theory, see e.g., [1, Proposition 6.3.16]. By a small perturbation of v_m, one can also assume that $\partial v_m / \partial n \neq 0$ along the boundary of each m-sim, unless v_m is constant there. □

We now return to the proof of Step 2. By Lemma 8, for each $n \in \mathbb{N}$, there exists $v_n \in R_k$ for some $k \in \mathbb{N}$ such that

$$\|v_n - u_n\|_{W^{1,1}(\mathbb{R}^d)} \le 1/n.$$

By Step 1, there exists a family $(v_{\delta,n}) \subset L^1(\mathbb{R}^d)$ such that $(v_{\delta,n})$ converges to v_n in $L^1(\mathbb{R}^d)$ as $\delta \to 0$ and

$$\limsup_{\delta \to 0} \Lambda_\delta(v_{\delta,n}) \le \kappa \int_{\mathbb{R}^d} |\nabla v_n| \, dx.$$

Hence there exists $\delta_n > 0$ such that, for $0 < \delta < \delta_n$

$$\Lambda_\delta(v_{\delta,n}) \le \kappa \int_{\mathbb{R}^d} |\nabla v_n| + 1/n \quad \text{and} \quad \|v_{\delta,n} - v_n\|_{L^1(\mathbb{R}^d)} \le 1/n.$$

Without loss of generality, one may assume that (δ_n) is decreasing to 0. Set

$$u_\delta = v_{\delta_{n+1},n} \quad \text{for } \delta_{n+1} \le \delta < \delta_n.$$

Then (u_δ) satisfies the properties required.

The proof of Case 1 is complete. □

Case 2: Ω is bounded. We prove Property (G2) for a general $u \in L^1(\Omega)$. Without loss of generality, one may assume that $u \in BV(\Omega)$. Let $R > 0$ be such that $\Omega \subset\subset B_R$ and let $(u_n) \subset C^\infty(\mathbb{R}^d)$ with $\operatorname{supp} u_n \subset B_R$ such that $u_n \to u$ in $L^1(\Omega)$ and $\|\nabla u_n\|_{L^1(\Omega)} \to \int_\Omega |\nabla u|$ as $n \to +\infty$ (the existence of such a sequence (u_n) is standard). Set, for $k \in \mathbb{N}$,

$$\Omega_k = \big\{x \in K \text{ for some } k\text{-sim } K \text{ such that } K \cap \Omega \neq \varnothing\big\}.$$

It is clear that, for each n,

$$\lim_{k \to +\infty} \int_{\Omega_k} |\nabla u_n| = \int_\Omega |\nabla u_n|. \tag{3.56}$$

By Lemma 8 (applied with $v = u_n$) and (3.56), for each n, there exist $k = k_n \in \mathbb{N}$ and $v_n \in R_k$ such that

$$\|v_n - u_n\|_{W^{1,1}(\mathbb{R}^d)} \le 1/n \quad \text{and} \quad \int_{\Omega_k} |\nabla v_n| \le \int_\Omega |\nabla v_n| + 1/n. \tag{3.57}$$

In what follows (except in the last two sentences), n is fixed. By Case 1 (applied with $u = v_n$), there exists a family $(v_{\delta,n}) \subset L^1(\mathbb{R}^d)$ such that $v_{\delta,n} \to v_n$ in $L^1(\mathbb{R}^d)$ as $\delta \to 0$ and

$$\limsup_{\delta \to 0} \Lambda_\delta(v_{\delta,n}, \mathbb{R}^d) \le \kappa \int_{\mathbb{R}^d} |\nabla v_n|. \tag{3.58}$$

Applying Lemma 6, we have

$$\liminf_{\delta \to 0} \Lambda_\delta(v_{\delta,n}, K) \geq \kappa \int_K |\nabla v_n| \text{ for each } k\text{-sim } K.$$

Since

$$\mathbb{R}^d \backslash \Omega_k = \bigcup_{\substack{K \text{ is a } k\text{-sim} \\ K \subset \mathbb{R}^d \backslash \Omega_k}} \bar{K},$$

it follows that

$$\liminf_{\delta \to 0} \Lambda_\delta(v_{\delta,n}, \mathbb{R}^d \backslash \Omega_k) \geq \kappa \int_{\mathbb{R}^d \backslash \Omega_k} |\nabla v_n|. \tag{3.59}$$

Clearly

$$\Lambda_\delta(v_{\delta,n}, \mathbb{R}^d \backslash \Omega_k) + \Lambda_\delta(v_{\delta,n}, \Omega_k) \leq \Lambda_\delta(v_{\delta,n}, \mathbb{R}^d). \tag{3.60}$$

We derive from (3.58), (3.59), and (3.60) that

$$\limsup_{\delta \to 0} \Lambda_\delta(v_{\delta,n}, \Omega_k) \leq \kappa \int_{\Omega_k} |\nabla v_n|,$$

which implies, by (3.57),

$$\limsup_{\delta \to 0} \Lambda_\delta(v_{\delta,n}, \Omega) \leq \kappa \int_\Omega |\nabla v_n| + \kappa/n.$$

Hence there exists $\delta_n > 0$ such that, for $0 < \delta < \delta_n$

$$\Lambda_\delta(v_{\delta,n}, \Omega) \leq \kappa \int_\Omega |\nabla v_n| + \kappa/n + 1/n \quad \text{and} \quad \|v_{\delta,n} - v_n\|_{L^1(\Omega)} \leq 1/n.$$

Without loss of generality, one may assume that (δ_n) is decreasing to 0. Set

$$u_\delta = v_{\delta_{n+1},n} \text{ in } \Omega \quad \text{for } \delta_{n+1} \leq \delta < \delta_n.$$

Then (u_δ) satisfies the required properties. □

3.4 Proof of Property (G1)

The starting point is again the definition of $\kappa(\varphi)$ given by (3.1) and (3.4), i.e.,

$$\kappa = \kappa(\varphi) = \inf \liminf_{\delta \to 0} \Lambda_\delta(v_\delta, \varphi, Q),$$

where the infimum is taken over all families of functions $(v_\delta)_{\delta \in (0,1)} \subset L^1(Q)$ such that $v_\delta \to U$ in $L^1(Q)$ as $\delta \to 0$.

The goal is to establish (G1) for every domain Ω, i.e., for every $u \in L^1(\Omega)$ and for every family $(u_\delta)_{\delta\in(0,1)} \subset L^1(\Omega)$ such that $u_\delta \to u$ in $L^1(\Omega)$ as $\delta \to 0$, one has

$$\liminf_{\delta\to 0} \Lambda_\delta(u_\delta, \varphi, \Omega) \geq \kappa \int_\Omega |\nabla u|.$$

It turns out to be convenient to replace U by another function (the function $H_{1/2}$ defined below) in the definition of κ. Set

$$H(x) = \begin{cases} 0 \text{ if } x_1 < 0, \\ 1 \text{ otherwise,} \end{cases}$$

and denote $H_c(x) := H(x_1 - c, x')$ for $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$ and $c \in \mathbb{R}$. Define

$$\gamma := \inf \liminf_{\delta\to 0} \Lambda_\delta(g_\delta, \varphi, Q), \tag{3.61}$$

where the infimum is taken over all families of functions $(g_\delta)_{\delta\in(0,1)} \subset L^1(Q)$ such that $g_\delta \to H_{1/2}$ in $L^1(Q)$. Note that $\int_Q |\nabla H_{1/2}| = 1$. It follows from Property (G2) that

$$\gamma \leq \kappa. \tag{3.62}$$

In the next section, we prove

Proposition 3 *We have*

$$\gamma = \kappa.$$

Clearly, this is consistent with Theorem 1.

The proof of (G1) in one dimension is based on Proposition 3 and the "essential variation" characterization of BV functions in one dimension (see, e.g., [33, Section 5.10.1]). The proof in higher dimensions is in the same spirit but much more involved. In order to be able to apply Proposition 3, we use Radon-Nikodym's theorem, a covering lemma à la Besicovitch, and a characterization of BV functions by slicing (see Sect. 3.4.2).

3.4.1 Proof of Proposition 3

The proof of Proposition 3 is based on two lemmas. The first one in the spirit of Lemma 4 is:

Lemma 9 *There exist a sequence $(h_k) \subset L^1(Q)$ and two sequences $(\delta_k), (c_k) \subset \mathbb{R}_+$ such that*

$$\lim_{k\to+\infty} \delta_k = \lim_{k\to+\infty} c_k = 0, \quad \lim_{k\to+\infty} h_k = H_{1/2} \textit{ in } L^1(Q),$$
$$h_k(x) = 0 \textit{ for } x_1 < 1/2 - c_k, \quad h_k(x) = 1 \textit{ for } x_1 > 1/2 + c_k, \quad 0 \leq h_k(x) \leq 1 \textit{ in } Q,$$

and

$$\lim_{k\to\infty} \Lambda_{\delta_k}(h_k, Q) = \gamma.$$

Proof From the definition of γ, there exist a sequence $(\tau_k) \subset \mathbb{R}_+$ and a sequence $(g_k) \subset L^1(Q)$ such that $\tau_k \to 0$, $g_k \to H_{\frac{1}{2}}$ in $L^1(Q)$, and

$$\lim_{k\to\infty} \Lambda_{\tau_k}(g_k, Q) = \gamma. \tag{3.63}$$

Set $c_k = \|g_k - H_{1/2}\|_{L^1(Q)}^{1/4}$ so that

$$\begin{aligned}\lim_{k\to+\infty} c_k = 0 \quad \text{and} \quad &\lim_{k\to\infty} \frac{\left|\{x \in Q; |g_k(x) - H_{\frac{1}{2}}(x)| \geq c_k\}\right|}{c_k} \\ &\leq \lim_{k\to\infty} \frac{\|g_k - H_{1/2}\|_{L^1(Q)}}{c_k^2} = 0.\end{aligned} \tag{3.64}$$

Define two continuous functions $h_{1,k}$, $h_{2,k} : Q \to \mathbb{R}$ which depend only on x_1 as follows

$$h_{1,k}(x) = \begin{cases} c_k & \text{if } x_1 < \frac{1}{2} - c_k, \\ 1 + c_k & \text{if } x_1 > \frac{1}{2}, \\ \text{affine w.r.t. } x_1 & \text{otherwise,} \end{cases}$$

$$h_{2,k}(x) = \begin{cases} -c_k & \text{if } x_1 < \frac{1}{2}, \\ 1 - c_k & \text{if } x_1 > \frac{1}{2} + c_k, \\ \text{affine w.r.t. } x_1 & \text{otherwise.} \end{cases}$$

Set

$$g_{1,k} = \min\big(\max(g_k, h_{2,k}), h_{1,k}\big) \quad \text{and} \quad g_{2,k} = \min\big(\max(g_{1,k}, c_k), 1 - c_k\big).$$

It is clear that, in Q,

$$\begin{aligned}&g_{2,k}(x) = c_k \text{ for } x_1 < 1/2 - c_k, \quad g_{2,k}(x) = 1 - c_k \text{ for } x_1 > 1/2 + c_k, \\ &c_k \leq g_{2,k} \leq 1 - c_k.\end{aligned} \tag{3.65}$$

We claim that

$$\limsup_{k\to\infty} \Lambda_{\tau_k}(g_{2,k}, Q) = \gamma. \tag{3.66}$$

Indeed, by Corollary 2, we have

$$\Lambda_{\tau_k}(g_{2,k}, Q) \leq \Lambda_{\tau_k}(g_{1,k}, Q). \tag{3.67}$$

Note that

$$\|\nabla h_{1,k}\|_{L^\infty(Q)} \le 1/c_k \quad \text{and} \quad \|\nabla h_{2,k}\|_{L^\infty(Q)} \le 1/c_k.$$

Using (3.64) and applying Lemma 2, we obtain

$$\limsup_{k\to\infty} \Lambda_{\tau_k}(g_{1,k}, Q) \le \limsup_{k\to\infty} \Lambda_{\tau_k}(g_k, Q). \tag{3.68}$$

Combining (3.63), (3.67), and (3.68) yields

$$\limsup_{k\to\infty} \Lambda_{\tau_k}(g_{2,k}, Q) \le \gamma;$$

which is (3.66). One can now verify that the conclusion holds for $h_k := (1 - 2c_k)^{-1}(g_{2,k} - c_k)$ and $\delta_k := (1 - 2c_k)^{-1}\tau_k$ by (3.65) and (3.66). $\square$

We next prove

Lemma 10 *Set $g(x) = x_1$ in Q. There exist a sequence $(g_k) \subset L^1(Q)$ and a sequence $(\delta_k) \subset \mathbb{R}_+$ such that*

$$\lim_{k\to+\infty} \delta_k = 0, \quad \lim_{k\to+\infty} g_k = g \textit{ in } L^1(Q),$$

and

$$\limsup_{k\to\infty} \Lambda_{\delta_k}(g_k, Q) \le \gamma.$$

Proof By Lemma 9, there exist a sequence $(h_k) \subset L^1(Q)$ and two sequences $(\delta_k), (c_k) \subset \mathbb{R}_+$ such that

$$\lim_{k\to+\infty} \delta_k = \lim_{k\to+\infty} c_k = 0, \tag{3.69}$$

$$h_k(x) = 0 \text{ for } x_1 < 1/2 - c_k, \quad h_k(x) = 1 \text{ for } x_1 > 1/2 + c_k,$$
$$0 \le h_k(x) \le 1 \text{ in } Q, \tag{3.70}$$

and

$$\lim_{k\to\infty} \Lambda_{\delta_k}(h_k, Q) = \gamma. \tag{3.71}$$

Fix $n \in \mathbb{N}$ and consider the sequence $(f_k) : Q \mapsto \mathbb{R}$ defined as follows

$$f_k(x) = \frac{1}{n} h_k\Big(x_1 - \frac{j}{n} + \frac{1}{2} - \frac{1}{2n}, x'\Big) + \frac{j}{n} \text{ for } x \in Q_j,\ 0 \le j \le n-1, \tag{3.72}$$

where $Q_j = [j/n, (j+1)/n] \times [0, 1]^{d-1}$. We deduce from (3.70) that

$$\int_Q |f_k(x) - x_1|\, dx \le \frac{1}{n}. \tag{3.73}$$

We claim that

$$\limsup_{k\to\infty} \Lambda_{\delta_k/n}(f_k, Q) \le \limsup_{k\to\infty} \Lambda_{\delta_k}(h_k, Q) = \gamma. \tag{3.74}$$

It is clear that

$$\Lambda_{\delta_k/n}(f_k, Q) \le \sum_{j=0}^{n-1} \Lambda_{\delta_k/n}(f_k, Q_j) + \sum_{j=0}^{n-1} \iint\limits_{Q_j\times(Q\setminus Q_j)} \frac{\varphi_{\delta_k/n}(|f_k(x)-f_k(y)|)}{|x-y|^{d+1}}\,dx\,dy. \tag{3.75}$$

Set $\hat{Q} = [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}] \times [0, 1]^{d-1}$. We have, by the definition of f_k,

$$\Lambda_{\delta_k/n}(f_k, Q_j) = \frac{1}{n}\Lambda_{\delta_k}(h_k, \hat{Q}) \le \frac{1}{n}\Lambda_{\delta_k}(h_k, Q). \tag{3.76}$$

If $(x, y) \in Q_j \times (Q\setminus Q_j)$ then $f_k(x) = f_k(y)$ if $|x_1 - y_1| < 1/(2n) - c_k$ by (3.70). It follows from (1.3) and (3.69) that

$$\begin{aligned} &\limsup_{k\to+\infty} \iint\limits_{Q_i\times(Q\setminus Q_i)} \frac{\varphi_{\delta_k/n}(|f_k(x)-f_k(y)|)}{|x-y|^{d+1}}\,dx\,dy \\ &\le \limsup_{k\to+\infty} \iint\limits_{\substack{Q_i\times(Q\setminus Q_i)\\ |x_1-y_1|\ge 1/(2n)-c_k}} \frac{b\delta_k/n}{|x-y|^{d+1}}\,dx\,dy = 0 \end{aligned} \tag{3.77}$$

(recall that n is fixed). Combining (3.71), (3.75), (3.76), and (3.77) yields (3.74).

We now reintroduce the dependence on n. By the above, there exists $f_{k,n}$, defined for $k, n \ge 1$, such that

$$\int_Q |f_{k,n}(x) - x_1|\,dx \le \frac{1}{n}$$

and

$$\limsup_{k\to\infty} \Lambda_{\delta_k/n}(f_{k,n}, Q) \le \gamma \text{ for each } n.$$

Thus for each n, there exists k_n such that $\Lambda_{\delta_{k_n}/n}(f_{k_n,n}, Q) \le \gamma + 1/n$. The desired conclusions hold for $(f_{k_n,n})$ and (δ_{k_n}/n). □

Proof of Proposition 3 We have $\kappa \le \gamma$ by Lemmas 6 and 10; and $\kappa \ge \gamma$ by (3.62). Hence $\gamma = \kappa$. □

3.4.2 Some Useful Lemmas

We begin with a consequence of the definition of γ and Proposition 3.

Lemma 11 *For any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\delta < \delta_\varepsilon$ and $g \in L^1(Q)$ with $\|g - H_{1/2}\|_{L^1(Q)} < \delta_\varepsilon$ then*

$$\Lambda_\delta(g, Q) \ge \kappa - \varepsilon.$$

We now prove

Lemma 12 *Let $c, \tau > 0$ and $(g_\delta) \subset L^1(\mathcal{R})$ with $\mathcal{R} = (a_1, b_1) \times (a, b)^{d-1}$ for some $a_1 < b_1$ and $a < b$ be such that $\tau < (b_1 - a_1)/8$. Assume that, for small δ,*

$$g_\delta(x) = 0 \text{ for } x_1 < a_1 + \tau, \quad g_\delta(x) = c \text{ for } x_1 > b_1 - \tau, \quad \text{and} \quad 0 \le g_\delta(x) \le c \text{ for } x \in \mathcal{R}.$$

We have, with $\mathcal{R}' = (a, b)^{d-1}$,

$$\liminf_{\delta \to 0} \Lambda_\delta(g_\delta, \mathcal{R}) \ge c\kappa |\mathcal{R}'|.$$

Here and in what follows, for a subset in $\mathbb{R}^{d-1}$, $|\cdot|$ denotes its $(d-1)$-dimensional Hausdorff measure unless stated otherwise.

Proof We only present the proof in two dimensions for simplicity of notations. Let $d = 2$. For $s > 0$, set

$$\mathcal{R}^s = (a_1, b_1) \times [(a, a+s) \cup (b-s, b)].$$

We first prove that, for every $s > 0$,

$$\liminf_{\delta \to 0} [\Lambda_\delta(g_\delta, \mathcal{R}) + \Lambda_\delta(g_\delta, \mathcal{R}^s)] \ge c\kappa |\mathcal{R}'|. \tag{3.78}$$

Without loss of generality, one may assume that

$$\mathcal{R} = (0, b_1) \times (0, b_2) \quad \text{and} \quad c = 1. \tag{3.79}$$

Let $g_{1,\delta} : (0, b_1) \times \mathbb{R}$ be such that

$$g_{1,\delta}(x) = g_\delta(x) \text{ for } x \in \mathcal{R}, \quad g_{1,\delta}(x) = g_\delta(x_1, -x_2) \text{ for } x \in (0, b_1) \times (-b_2, 0), \tag{3.80}$$

and $g_{1,\delta}$ is a periodic function in x_2 with period $2b_2$. Set

$$\mathcal{R}_j = (0, b_1) \times (jb_2, jb_2 + b_2) \text{ for } j \ge 0 \quad \text{and} \quad \mathcal{R}(m) = (0, b_1) \times (0, 2mb_2) \text{ for } m \ge 0.$$

It is clear that, for $m \in \mathbb{N}$,

$$\Lambda_\delta(g_{1,\delta}, \mathcal{R}(m)) = \sum_{j=0}^{2m-1} \Lambda_\delta(g_{1,\delta}, \mathcal{R}_j) + \sum_{j=0}^{2m-1} \iint_{\mathcal{R}_j \times (\mathcal{R}(m) \setminus \mathcal{R}_j)} \frac{\varphi_\delta(|g_{1,\delta}(x) - g_{1,\delta}(y)|)}{|x-y|^3} \, dx \, dy. \tag{3.81}$$

From the definition of $g_{1,\delta}$, we have, for $0 \le j \le 2m-1$,

$$\Lambda_\delta(g_{1,\delta}, \mathcal{R}_j) = \Lambda_\delta(g_\delta, \mathcal{R}). \tag{3.82}$$

Clearly, for $0 \le j \le 2m - 1$,

$$\iint\limits_{\mathcal{R}_j \times (\mathcal{R}(m) \setminus \mathcal{R}_j)} \frac{\varphi_\delta(|g_{1,\delta}(x) - g_{1,\delta}(y)|)}{|x-y|^3}\, dx\, dy = \iint\limits_{\substack{\mathcal{R}_j \times (\mathcal{R}(m) \setminus \mathcal{R}_j) \\ |x_2 - y_2| < s}} \frac{\varphi_\delta(|g_{1,\delta}(x) - g_{1,\delta}(y)|)}{|x-y|^3}\, dx\, dy + \iint\limits_{\substack{\mathcal{R}_j \times (\mathcal{R}(m) \setminus \mathcal{R}_j) \\ |x_2 - y_2| \ge s}} \frac{\varphi_\delta(|g_{1,\delta}(x) - g_{1,\delta}(y)|)}{|x-y|^3}\, dx\, dy;$$

which yields, by the definition of $g_{1,\delta}$ and (1.3),

$$\iint\limits_{\mathcal{R}_j \times (\mathcal{R}(m) \setminus \mathcal{R}_j)} \frac{\varphi_\delta(|g_{1,\delta}(x) - g_{1,\delta}(y)|)}{|x-y|^3}\, dx\, dy \le \Lambda_\delta(g_\delta, \mathcal{R}^s) + C\delta m / s^3. \tag{3.83}$$

Here and in what follows in this proof, C denotes a positive constant independent of δ and m. Combining (3.81), (3.82), and (3.83) yields

$$\Lambda_\delta(g_{1,\delta}, \mathcal{R}(m)) \le 2m \Lambda_\delta(g_\delta, \mathcal{R}) + 2m \Lambda_\delta(g_\delta, \mathcal{R}^s) + C\delta m^2 / s^3. \tag{3.84}$$

Define $g_{2,\delta} : \mathbb{R}^2 \to \mathbb{R}$ as follows

$$g_{2,\delta}(x) = \begin{cases} g_{1,\delta}(x) & \text{if } x_1 \in (0, b_1), \\ 0 & \text{if } x_1 \le 0, \\ 1 & \text{if } x_1 \ge b_1. \end{cases}$$

Since $g_{1,\delta}(x) = 0$ for $x_1 < \tau$ and $g_{1,\delta}(x) = 1$ for $x_1 > b_1 - \tau$, by (1.3), we have, for $m \in \mathbb{N}$,

$$\Lambda_\delta(g_{2,\delta}, (-mb_2, mb_2) \times (0, 2mb_2)) \le \Lambda_\delta(g_{1,\delta}, \mathcal{R}(m)) + C\delta m^4 / \tau^3. \tag{3.85}$$

Set, for $m \in \mathbb{N}$ and $x \in Q$,

$$g_{3,\delta,m}(x) = g_{2,\delta}\big(2b_2 m(x_1 - 1/2, x_2)\big).$$

By a change of variables, we have

$$\Lambda_\delta(g_{2,\delta}, (-mb_2, mb_2) \times (0, 2mb_2)) = 2mb_2 \Lambda_\delta(g_{3,\delta,m}, Q). \tag{3.86}$$

Combining (3.84), (3.85), and (3.86) yields

$$b_2 \Lambda_\delta(g_{3,\delta,m}, Q) \le \Lambda_\delta(g_\delta, \mathcal{R}) + \Lambda_\delta(g_\delta, \mathcal{R}^s) + C\delta m / s^3 + C\delta m^3 / \tau^3. \tag{3.87}$$

Since $0 \le g_{2,\delta} \le 1$, it follows from the definition of $g_{3,\delta,m}$ that

$$\|g_{3,\delta,m} - H_{1/2}\|_{L^1(Q)} \le C/m.$$

By Lemma 11, for every $\varepsilon > 0$ there exists $m_\varepsilon > 0$ such that if $m \geq m_\varepsilon$ then

$$\liminf_{\delta \to 0} \Lambda_\delta(g_{3,\delta,m}, Q) \geq \kappa - \varepsilon. \tag{3.88}$$

Taking $m = m_\varepsilon$ in (3.87), we derive from (3.88) that

$$\liminf_{\delta \to 0} [\Lambda_\delta(g_\delta, \mathcal{R}) + \Lambda_\delta(g_\delta, \mathcal{R}^s)] \geq (\kappa - \varepsilon) b_2. \tag{3.89}$$

Since $\varepsilon > 0$ is arbitrary, we obtain (3.78) by (3.79).

We are now ready to prove

$$\liminf_{\delta \to 0} \Lambda_\delta(g_\delta, \mathcal{R}) \geq c\kappa |\mathcal{R}'|. \tag{3.90}$$

Without loss of generality, one may again assume (3.79). Fix $n \in \mathbb{N}$ (arbitrary) and define $s = s(n) = b_2/(4n^2)$. For $0 \leq j \leq n$, by (3.78) (applied with $\mathcal{R} = \mathcal{R} \backslash \mathcal{R}^{js}$), we have,

$$\liminf_{\delta \to 0} \left[\Lambda_\delta(g_\delta, \mathcal{R} \backslash \mathcal{R}^{js}) + \Lambda_\delta(g_\delta, \mathcal{R}^{js+s} \backslash \mathcal{R}^{js}) \right] \geq \kappa(b_2 - 2js).$$

Summing these inequalities for j from 0 to $n-1$ and noting that

$$\Lambda_\delta(g_\delta, \mathcal{R}) \geq \sum_{j=0}^{n-1} \Lambda_\delta(g_\delta, \mathcal{R}^{js+s} \backslash \mathcal{R}^{js}) \quad \text{and} \quad b_2 - 2js \geq b_2 - b_2/(2n),$$

we obtain

$$\liminf_{\delta \to 0} (n+1) \Lambda_\delta(g_\delta, \mathcal{R}) \geq n\kappa b_2 [1 - 1/(2n)].$$

This implies

$$\liminf_{\delta \to 0} \Lambda_\delta(g_\delta, \mathcal{R}) \geq \frac{n}{n+1} \kappa b_2 [1 - 1/(2n)].$$

Since $n \in \mathbb{N}$ is arbitrary, we obtain

$$\liminf_{\delta \to 0} \Lambda_\delta(g_\delta, \mathcal{R}) \geq \kappa b_2.$$

The proof is complete. $\square$

Here is a more general version of Lemma 12.

Lemma 13 *Let $c, \tau > 0$ and $(g_\delta) \subset L^1(\mathcal{R})$ with $\mathcal{R} = (a_1, b_1) \times (a, b)^{d-1}$ for some $a_1 < b_1$ and $a < b$ be such that $\tau < (b_1 - a_1)/8$. Set*

$$A_\delta = \left\{ x \in \mathcal{R} : g_\delta(x) > 0 \text{ and } a_1 \leq x_1 \leq a_1 + \tau \right\} \text{ and}$$
$$B_\delta = \left\{ x \in \mathcal{R} : \ g_\delta(x) < c \text{ and } b_1 - \tau \leq x_1 \leq b_1 \right\}.$$

We have, with $\mathcal{R}' = (a, b)^{d-1}$,

$$\liminf_{\delta \to 0} \Lambda_\delta(g_\delta, \mathcal{R}) \geq c\kappa |\mathcal{R}'| - C_d c \limsup_{\delta \to 0} (|A_\delta| + |B_\delta|)/\tau,$$

Proof Define two continuous functions f_1 and f_2 in $\mathcal{R}$ which depend only on x_1 as follows

$$f_1(x) = \begin{cases} 0 & \text{if } x_1 \leq a_1 + \tau/2, \\ c & \text{if } x_1 \geq a_1 + \tau, \\ \text{affine w.r.t. } x_1 & \text{otherwise,} \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 & \text{if } x_1 \leq b_1 - \tau, \\ c & \text{if } x_1 \geq b_1 - \tau/2, \\ \text{affine w.r.t. } x_1 & \text{otherwise.} \end{cases}$$

Define

$$h_{1,\delta} = \max\Big(\min\big(g_\delta, c\big), 0\Big), \quad h_{2,\delta} = \min(h_{1,\delta}, f_1), \quad \text{and} \quad h_{3,\delta} = \max(h_{2,\delta}, f_2).$$

By Corollary 2, we have

$$\Lambda_\delta(h_{1,\delta}, \mathcal{R}) \leq \Lambda_\delta(g_\delta, \mathcal{R}),$$

and by Lemma 2, we obtain

$$\Lambda_\delta(h_{2,\delta}, \mathcal{R}) \leq \Lambda_\delta(h_{1,\delta}, \mathcal{R}) + C_d c |\{h_{1,\delta} > f_1\}|/\tau \leq \Lambda_\delta(h_{2,\delta}, \mathcal{R}) + C_d c |A_\delta|/\tau,$$

and

$$\Lambda_\delta(h_{3,\delta}, \mathcal{R}) \leq \Lambda_\delta(h_{2,\delta}, \mathcal{R}) + C_d c |\{h_{2,\delta} < f_2\}|/\tau \leq \Lambda_\delta(h_{2,\delta}, \mathcal{R}) + C_d c |B_\delta|/\tau.$$

It follows that

$$\Lambda_\delta(h_{3,\delta}, \mathcal{R}) \leq \Lambda_\delta(g_\delta, \mathcal{R}) + C_d c(|A_\delta| + |B_\delta|)/\tau.$$

One can easily check that $h_{3,\delta}(x) = 0$ for $x_1 \leq a_1 + \tau/2$, $h_{3,\delta}(x) = c$ for $x_1 \geq b_1 - \tau/2$, and $0 \leq h_{3,\delta} \leq c$ in $\mathcal{R}$. Applying Lemma 12 for $h_{3,\delta}$, we obtain the conclusion. $\square$

We next recall the definition of a Lebesgue surface (see [45]):

Definition 2 Let $g \in L^1(\mathcal{R})$ with $\mathcal{R} = \prod_{i=1}^d (a_i, b_i)$ for some $a_i < b_i$ $(1 \leq i \leq d)$ and $t \in (a_1, b_1)$. Set $\mathcal{R}' = \prod_{i=2}^d (a_i, b_i)$. The surface $x_1 = t$ is said to be a Lebesgue surface of g if for almost every $z' \in \mathcal{R}'$, (t, z') is a Lebesgue point of g, the restriction of g to the surface $x_1 = t$ is integrable with respect to $(d-1)$-Hausdorff measure, and

$$\lim_{\varepsilon \to 0_+} \fint_{t-\varepsilon}^{t+\varepsilon} \int_{\mathcal{R}'} |g(s, z') - g(t, z')| \, dz' \, ds = 0. \tag{3.91}$$

For $i = 2, \dots, d$, we also define the notion of the Lebesgue surface for surfaces $x_i = t$ with $t \in (a_i, b_i)$ in a similar manner.

The following lemma plays a crucial role in our analysis; its proof relies on Lemma 13.

Lemma 14 *Let $g \in L^1(\mathcal{R})$ and $(g_\delta) \subset L^1(\mathcal{R})$ with $\mathcal{R} = \prod_{i=1}^d (a_i, b_i)$ for some $a_i < b_i$ ($1 \le i \le d$) such that $(g_\delta) \to g$ in $L^1(\mathcal{R})$. Set $\mathcal{R}' = \prod_{i=2}^d (a_i, b_i)$. Let $a_1 < t_1 < t_2 < b_1$ be such that the surface $x_1 = t_j$ ($j = 1, 2$) is a Lebesgue surface of g. We have*

$$\liminf_{\delta \to 0} \Lambda_\delta\big(g_\delta, (t_1, t_2) \times \mathcal{R}'\big) \ge \kappa \int\limits_{\mathcal{R}'} |g(t_2, x') - g(t_1, x')| \, dx'.$$

Proof Fix $\varepsilon > 0$ (arbitrary). Let A' be the set of all elements $z' \in \mathcal{R}'$ such that, for $j = 1, 2$, (t_j, z') is a Lebesgue point of $g(t_j, \cdot)$ and (t_j, z') is a Lebesgue point of g. Then $|A'| = |\mathcal{R}'|$. For each $z' \in A'$, there exists $l(z', \varepsilon) > 0$ such that for every closed cube $Q_l'(z') \subset \mathbb{R}^{d-1}$ centered at z' with length $0 < l < l(z', \varepsilon)$, we have

$$\begin{cases} \Big|\big\{(x_1, y') \in (t_1, t_1 + l) \times Q_l'(z');\ |g(x_1, y') - g(t_1, z')| \ge \varepsilon/2\big\}\Big| \le \varepsilon l^d, \\ \Big|\big\{(x_1, y') \in (t_2 - l, t_2) \times Q_l'(z');\ |g(x_1, y') - g(t_2, z')| \ge \varepsilon/2\big\}\Big| \le \varepsilon l^d, \end{cases} \tag{3.92}$$

and, for $j = 1, 2$,

$$\fint_{Q_l'(z')} |g(t_j, y') - g(t_j, z')| \, dy' < \varepsilon. \tag{3.93}$$

Fix $z' \in A'$ and $0 < l < l(z', \varepsilon)$. Since $(g_\delta) \to g$ in $L^1(\mathcal{R})$, it follows from (3.92) that, when δ is small,

$$\begin{cases} \Big|\big\{(x_1, y') \in (t_1, t_1 + l) \times Q_l'(z');\ |g_\delta(x_1, y') - g(t_1, z')| \ge \varepsilon\big\}\Big| \le 2\varepsilon l^d, \\ \Big|\big\{(x_1, y') \in (t_2 - l, t_2) \times Q_l'(z');\ |g_\delta(x_1, y') - g(t_2, z')| \ge \varepsilon\big\}\Big| \le 2\varepsilon l^d. \end{cases} \tag{3.94}$$

We claim that

$$\liminf_{\delta \to 0} \Lambda_\delta\big(g_\delta, (t_1, t_2) \times Q_l'(z')\big) \ge \kappa \big|g(t_2, z') - g(t_1, z')\big| \, |Q_l'(z')| - C\varepsilon |Q_l'(z')|, \tag{3.95}$$

for some positive constant C depending only on d. Indeed, without loss of generality, one may assume that $g(t_2, z') \ge g(t_1, z')$. It is clear that (3.95) holds if $g(t_2, z') - g(t_1, z') \le 4\varepsilon$ by choosing $C = 10$. We now consider the case $g(t_2, z') - g(t_1, z') > 4\varepsilon$. Applying Lemma 13 for $g_\delta - g(t_1, z') - \varepsilon$ in the set $(t_1, t_2) \times Q_l'(z')$, $\tau = l$, and $c = g(t_2, z') - g(t_1, z') - 2\varepsilon$, we derive that

$$\liminf_{\delta \to 0} \Lambda_\delta\big(g_\delta, (t_1, t_2) \times Q_l'(z')\big) \ge \kappa \big|g(t_2, z') - g(t_1, z') - 2\varepsilon\big| \, |Q_l'(z')| - C\varepsilon |Q_l'(z')|$$

since, by (3.94),

$$\begin{aligned}|A_\delta| &= \Big|\big\{(x_1, y') \in (t_1, t_1 + l) \times Q'_l(z');\ g_\delta(x_1, y') - g(t_1, z') - \varepsilon > 0\big\}\Big| \\ &\le \Big|\big\{(x_1, y') \in (t_1, t_1 + l) \times Q'_l(z');\ |g_\delta(x_1, y') - g(t_1, z')| \\ &\ge \varepsilon\big\}\Big| \le 2\varepsilon l^d = 2\varepsilon l |Q'_l(z')|\end{aligned}$$

and

$$\begin{aligned}|B_\delta| &= \Big|\big\{(x_1, y') \in (t_2 - l, t_2) \times Q'_l(z');\ g_\delta(x_1, y') - g(t_1, z') \\ &\qquad - 2\varepsilon < g(t_2, z') - g(t_1, z') - \varepsilon\big\}\Big| \\ &\le \Big|\big\{(x_1, y') \in (t_1, t_1 + l) \times Q'_l(z');\ |g(x_1, y') - g(t_1, z')| \ge \varepsilon\big\}\Big| \\ &\le 2\varepsilon l^d = 2\varepsilon l |Q'_l(z')|.\end{aligned}$$

This implies Claim (3.95).

On the other hand, by Besicovitch's covering theorem (see e.g., [33, Corollary 1 on page 35][1]), there exist a sequence $(z'_k)_{k\in\mathbb{N}} \subset A'$ and disjoint cubes $\big(Q'_{l_k}(z'_k)\big)_{k\in\mathbb{N}} \subset \mathcal{R}'$ such that $0 < l_k < l(z'_k, \varepsilon)$ for every k, and

$$|A'| = \sum_{k\in\mathbb{N}} |Q'_{l_k}(z'_k)|. \tag{3.96}$$

[For the convenience of the reader, we explain how to apply [33, Corollary 1] in our situation. We take $n = d-1$, A is our A', $\mathcal{F} = \big\{Q'_l(z');\ z' \in A' \text{ and } 0 < l < l(z', \varepsilon)\big\}$, and $U = \mathcal{R}'$.]

It follows from (3.95) that

$$\liminf_{\delta\to 0} \Lambda_\delta\big(g_\delta, (t_1, t_2)\times\mathcal{R}'\big) \ge \sum_k \Big(\kappa\big|g(t_2, z'_k) - g(t_1, z'_k)\big|\,|Q'_{l_k}(z'_k)| - C\varepsilon|Q'_{l_k}(z'_k)|\Big). \tag{3.97}$$

We claim that

$$\big|g(t_2, z'_k) - g(t_1, z'_k)\big|\,|Q'_{l_k}(z'_k)| \ge \int_{Q'_{l_k}(z'_k)} |g(t_2, y') - g(t_1, y')|dy' - 2\varepsilon|Q'_{l_k}(z'_k)|. \tag{3.98}$$

Indeed, we have

$$\begin{aligned}\int_{Q'_{l_k}(z'_k)} |g(t_2, y') - g(t_1, y')|dy' &\le \int_{Q'_{l_k}(z'_k)} |g(t_2, y') - g(t_2, z')| + |g(t_1, y') - g(t_1, z')|dy' \\ &\quad + \big|g(t_2, z'_k) - g(t_1, z'_k)\big|\,|Q'_{l_k}(z'_k)|;\end{aligned}$$

[1] In [33], this result is stated for balls but a similar argument works for cubes with arbitrary orientations.

which implies (3.98) by (3.93).

Combining (3.96), (3.97), and (3.98) yields

$$\liminf_{\delta \to 0} \Lambda_\delta\big(g_\delta, (t_1, t_2) \times \mathcal{R}'\big) \geq \kappa \int_{\mathcal{R}'} |g(t_2, y') - g(t_1, y')| dy' - C\varepsilon |\mathcal{R}'|.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the conclusion. □

We next recall the notion of essential variation in [45] related to BV functions.

Definition 3 Let $g \in L^1(\mathcal{R})$ with $\mathcal{R} = \prod_{i=1}^d (a_i, b_i)$ for some $a_i < b_i$ $(1 \leq i \leq d)$. Set $\mathcal{R}' = \prod_{i=2}^d (a_i, b_i)$. The essential variation of g in the first direction is defined as follows

$$\operatorname{ess} V(g, 1, \mathcal{R}) = \sup \left\{ \sum_{i=1}^m \int_{\mathcal{R}'} |g(t_{i+1}, x') - g(t_i, x')|\, dx' \right\},$$

where the supremum is taken over all finite partitions $\{a_1 < t_1 < \cdots < t_{m+1} < b_1\}$ such that the surface $x_1 = t_k$ is a Lebesgue surface of g for $1 \leq k \leq m + 1$. For $2 \leq j \leq d$, we also define $\operatorname{ess} V(g, j)$ the essential variation of g in the j^{th} direction in a similar manner.

The following result provides a characterization of BV functions (see e.g., [45, Proposition 3]).

Proposition 4 *Let $g \in L^1(\mathcal{R})$ with $\mathcal{R} = \prod_{i=1}^d (a_i, b_i)$ for some $a_i < b_i$ $(1 \leq i \leq d)$. Then $g \in BV(\mathcal{R})$ if and only if*

$$\operatorname{ess} V(g, j, \mathcal{R}) < +\infty, \quad \forall\, 1 \leq j \leq d.$$

Moreover, for $g \in BV(\mathcal{R})$,

$$\operatorname{ess} V(g, j, \mathcal{R}) = \int_{\mathcal{R}} \left| \frac{\partial g}{\partial x_j} \right| \quad \forall\, 1 \leq j \leq d.$$

As a consequence of Lemma 14 and Proposition 4, we have

Corollary 6 *Let $g \in L^1(\mathcal{R})$ and $(g_\delta) \subset L^1(\mathcal{R})$ with $\mathcal{R} = \prod_{i=1}^d (a_i, b_i)$ for some $a_i < b_i$ $(1 \leq i \leq d)$ such that $(g_\delta) \to g$ in $L^1(\mathcal{R})$. We have*

$$\liminf_{\delta \to 0} \Lambda_\delta\big(g_\delta, \mathcal{R}\big) \geq \kappa \int_{\mathcal{R}} \left| \frac{\partial g}{\partial x_j} \right| \quad \forall\, 1 \leq j \leq d.$$

3.4.3 Proof of Property (G1) Completed

Recall that for each $u \in BV(\Omega)$, $|\nabla u|$ is a Radon measure on Ω. By Radon-Nikodym's theorem, we may write

$$\nabla u = \sigma |\nabla u|,$$

for some $\sigma \in L^\infty(\Omega, |\nabla u|, \mathbb{R}^d)$ and $|\sigma| = 1$ $|\nabla u|$- a.e. (see e.g., [33, Theorem 1 on page 167]). Then for $|\nabla u|$- a.e. $x \in \Omega$, one has (see e.g., [33, Theorem 1 on page 43])

$$\lim_{r\to 0} \frac{1}{|\nabla u|(Q_r(x, \sigma(x)))} \int_{Q_r(x,\sigma(x))} \sigma(y)|\nabla u(y)|\,dy = \sigma(x)^2.$$

Hereafter for any $(x, \sigma, r) \in \Omega \times \mathbb{S}^{d-1} \times (0, +\infty)$, $Q_r(x, \sigma)$ denotes the closed cube centered at x with edge length $2r$ such that one of its faces is orthogonal to σ. It follows that, for $|\nabla u|$- a.e. $x \in \Omega$,

$$\lim_{r\to 0} \frac{1}{|\nabla u|(Q_r(x, \sigma(x)))} \int_{Q_r(x,\sigma(x))} \sigma(y)\cdot\sigma(x)|\nabla u(y)|\,dy = 1.$$

Since

$$\sigma(y)\cdot\sigma(x) \le |\sigma(y)\cdot\sigma(x)| \le 1,$$

we derive that, for $|\nabla u|$- a.e. $x \in \Omega$,

$$\lim_{r\to 0} \frac{1}{|\nabla u|(Q_r(x, \sigma(x)))} \int_{Q_r(x,\sigma(x))} |\sigma(y)\cdot\sigma(x)|\,|\nabla u(y)|\,dy = 1.$$

In other words, for $|\nabla u|$- a.e. $x \in \Omega$,

$$\lim_{r\to 0} \int_{Q_r(x,\sigma(x))} |\nabla u(y)\cdot\sigma(x)|\,dy \Big/ \int_{Q_r(x,\sigma(x))} |\nabla u(y)|\,dy = 1. \tag{3.99}$$

Denote $A = \{x \in \Omega;\ (3.99) \text{ holds}\}$. Fix $\varepsilon > 0$ (arbitrary). For $x \in A$, there exists a sequence $s_n = s_n(x, \varepsilon) \to 0$ as $n \to +\infty$ such that, for all n,

$$\int_{Q_{s_n}(x,\sigma(x))} |\nabla u(y)\cdot\sigma(x)|\,dy \Big/ \int_{Q_{s_n}(x,\sigma(x))} |\nabla u(y)|\,dy \ge 1-\varepsilon. \tag{3.100}$$

and

$$\int_{\partial Q_{s_n}(x,\sigma(x))} |\nabla u(y)|\,dy = 0. \tag{3.101}$$

Set

$$\mathcal{F} = \big\{Q_{s_n(x,\varepsilon)}(x, \sigma(x));\ x \in A \text{ and } n \in \mathbb{N}\big\}.$$

By Besicovitch's covering theorem (see e.g., [33, Corollary 1 on page 35] applied with A, $\mathcal{F}$, $U = \Omega$, and $\mu = |\nabla u|$), there exists a collection of disjoint cubes $\Big(Q_{r_k}\big(x_k, \sigma(x_k)\big)\Big)_{k\in\mathbb{N}}$ with $x_k \in A$ and $r_k = s_{n_k}(x_k, \varepsilon)$ such that

[2] In [33], this result is stated for balls but a similar argument works for cubes with arbitrary orientations.

$$|\nabla u|(\Omega) = |\nabla u|\Big(\bigcup_{k \in \mathbb{N}} Q_{r_k}\big(x_k, \sigma(x_k)\big)\Big). \tag{3.102}$$

From (3.100) and (3.101), we have

$$\int_{Q_{r_k}(x_k, \sigma(x_k))} |\nabla u(y)| \, dy \le \frac{1}{1-\varepsilon} \int_{Q_{r_k}(x_k, \sigma(x_k))} |\nabla u(y) \cdot \sigma(x_k)| \, dy \tag{3.103}$$

and

$$\int_{\partial Q_{r_k}(x_k, \sigma(x_k))} |\nabla u(y)| \, dy = 0. \tag{3.104}$$

Combining (3.102) and (3.103) yields

$$|\nabla u|(\Omega) \le \frac{1}{1-\varepsilon} \sum_{k \in \mathbb{N}} \int_{Q_{r_k}(x_k, \sigma(x_k))} |\nabla u(y) \cdot \sigma(x_k)| \, dy. \tag{3.105}$$

Applying Corollary 6 and using (3.104), we obtain

$$\kappa \int_{Q_{r_k}(x_k, \sigma(x_k))} |\nabla u(y) \cdot \sigma(x_k)| \, dy \le \liminf_{\delta \to 0} \Lambda_\delta\big(u_\delta, Q_{r_k}(x_k, \sigma(x_k))\big). \tag{3.106}$$

From (3.105) and (3.106), we have

$$\kappa |\nabla u|(\Omega) \le \frac{1}{1-\varepsilon} \liminf_{\delta \to 0} \Lambda_\delta\big(u_\delta, \Omega\big). \tag{3.107}$$

Since $\varepsilon > 0$ is arbitrary, we have established that, for $u \in BV(\Omega)$,

$$\liminf_{\delta \to 0} \Lambda_\delta\big(u_\delta, \Omega\big) \ge \kappa |\nabla u|(\Omega).$$

Suppose now that $u \in BV_{loc}(\Omega)$, we may apply the above for any $\omega \subset\subset \Omega$ and therefore we conclude that

$$\liminf_{\delta \to 0} \Lambda_\delta\big(u_\delta, \Omega\big) \ge \kappa |\nabla u|(\Omega).$$

Hence it now suffices to prove that if $\liminf_{\delta \to} \Lambda_\delta(u_\delta, \Omega) < +\infty$, then $u \in BV_{loc}(\Omega)$. Indeed, this is a consequence of Corollary 6.

The proof is complete. □

3.5 Further Properties of $K(\varphi)$

This section deals with properties of $\kappa(\varphi)$ defined in (3.4). Our main result is:

Theorem 5 *We have*

$$\kappa(\varphi) \ge \kappa(c_1\tilde{\varphi}_1) \quad \textit{for all } \varphi \in \mathcal{A}; \tag{3.108}$$

in particular,

$$\inf_{\varphi \in \mathcal{A}} \kappa(\varphi) > 0.$$

Proof The proof uses an idea in [44, Section 2.3]. From the definition of $\kappa(c_1\tilde{\varphi}_1)$, we have (see [45, Lemma 8])

$$\begin{aligned}&\forall\, \varepsilon > 0,\ \exists \delta(\varepsilon) > 0 \text{ such that if } \|v - U\|_{L^1(Q)} < \delta(\varepsilon) \text{ and}\\ &\delta < \delta(\varepsilon),\ \text{then } \Lambda_\delta(v, c_1\tilde{\varphi}_1) \ge \kappa(c_1\tilde{\varphi}_1) - \varepsilon.\end{aligned} \tag{3.109}$$

Next we **fix** $\varphi \in \mathcal{A}$. **Fix** $(u_\delta) \subset L^1(Q)$ be such that $u_\delta \to U$ in $L^1(Q)$. Our goal is to prove that

$$\liminf_{\delta \to 0} \Lambda_\delta(u_\delta, \varphi) \ge \kappa(c_1\tilde{\varphi}_1). \tag{3.110}$$

Let $c > 1$ and $\varepsilon > 0$. Since φ is non-decreasing, we have

$$\int_Q \int_Q \frac{\varphi_\delta(|u_\delta(x) - u_\delta(y)|)}{|x-y|^{d+1}}\,dxdy \ge \sum_{k=-\infty}^{\infty} \underset{c^{-k-1} < |u_\delta(x) - u_\delta(y)| \le c^{-k}}{\int_Q \int_Q} \frac{\varphi_\delta(c^{-k-1})}{|x-y|^{d+1}}\,dxdy. \tag{3.111}$$

Using the fact that

$$\begin{aligned}&\underset{c^{-k-1} < |u_\delta(x) - u_\delta(y)| \le c^{-k}}{\int_Q \int_Q} \frac{1}{|x-y|^{d+1}}\,dxdy\\ &\quad = \underset{|u_\delta(x) - u_\delta(y)| > c^{-k-1}}{\int_Q \int_Q} \frac{1}{|x-y|^{d+1}}\,dxdy - \underset{|u_\delta(x) - u_\delta(y)| > c^{-k}}{\int_Q \int_Q} \frac{1}{|x-y|^{d+1}}\,dxdy,\end{aligned}$$

we obtain

$$\begin{aligned}&\sum_{k=-\infty}^{\infty} \underset{c^{-k-1} < |u_\delta(x) - u_\delta(y)| \le c^{-k}}{\int_Q \int_Q} \frac{\varphi_\delta(c^{-k-1})}{|x-y|^{d+1}}\,dxdy\\ &\quad = \sum_{k=-\infty}^{\infty} \left[\varphi_\delta(c^{-k}) - \varphi_\delta(c^{-k-1})\right] \underset{|u_\delta(x) - u_\delta(y)| > c^{-k}}{\int_Q \int_Q} \frac{1}{|x-y|^{d+1}}\,dxdy.\end{aligned} \tag{3.112}$$

We have, for any $k_0 > 0$,

$$\sum_{k=-\infty}^{\infty} \big[\varphi_\delta(c^{-k}) - \varphi_\delta(c^{-k-1})\big] \mathop{\int_Q \int_Q}_{|u_\delta(x)-u_\delta(y)|>c^{-k}} \frac{1}{|x-y|^{d+1}}\,dxdy$$
$$\geq \sum_{k=k_0}^{\infty} \big[\varphi_\delta(c^{-k}) - \varphi_\delta(c^{-k-1})\big] c^k \mathop{\int_Q \int_Q}_{|u_\delta(x)-u_\delta(y)|>c^{-k}} \frac{c^{-k}}{|x-y|^{d+1}}\,dxdy. \tag{3.113}$$

Applying (3.109) with $v = u_\delta$ and $\delta = c^{-k}$, we obtain

$$c_1 \mathop{\int_Q \int_Q}_{|u_\delta(x)-u_\delta(y)|>c^{-k}} \frac{c^{-k}}{|x-y|^{d+1}}\,dxdy \geq \kappa(c_1\tilde{\varphi}_1) - \varepsilon, \tag{3.114}$$

provided $\|u_\delta - U\|_{L^1(Q)} < \delta(\varepsilon)$ and $c^{-k} < \delta(\varepsilon)$. In particular, there exist $\tilde{\delta}(\varepsilon) > 0$ and $k(\varepsilon, c) \in \mathbb{N}$ such that (3.114) holds for $\delta < \tilde{\delta}(\varepsilon)$ and $k \geq k(\varepsilon, c)$. Combining (3.113) and (3.114) yields

$$\sum_{k=-\infty}^{\infty} \big[\varphi_\delta(c^{-k}) - \varphi_\delta(c^{-k-1})\big] \mathop{\int_Q \int_Q}_{|u_\delta(x)-u_\delta(y)|>c^{-k}} \frac{1}{|x-y|^{d+1}}\,dxdy$$
$$\geq c_1^{-1} \sum_{k_0}^{\infty} \big[\kappa(c_1\tilde{\varphi}_1) - \varepsilon\big] c^k \big[\varphi_\delta(c^{-k}) - \varphi_\delta(c^{-k-1})\big], \tag{3.115}$$

for $k_0 = k(\varepsilon, c)$ and $\delta < \tilde{\delta}(\varepsilon)$. We derive from (3.111), (3.112), and (3.115) that, for $\delta < \tilde{\delta}(\varepsilon)$,

$$\Lambda_\delta(u_\delta) \geq c_1^{-1}\big[\kappa(c_1\tilde{\varphi}_1) - \varepsilon\big] \sum_{k=k_0}^{\infty} c^k \big[\varphi_\delta(c^{-k}) - \varphi_\delta(c^{-k-1})\big]. \tag{3.116}$$

We have, since $\varphi_\delta \geq 0$,

$$\sum_{k=k_0}^{\infty} c^k \big[\varphi_\delta(c^{-k}) - \varphi_\delta(c^{-k-1})\big] = \sum_{k=k_0}^{\infty} c^k \varphi_\delta(c^{-k}) - \sum_{k=k_0}^{\infty} c^k \varphi_\delta(c^{-k-1})$$
$$\geq \frac{1}{c} \sum_{k=k_0+1}^{\infty} \varphi_\delta(c^{-k}) c^k (c-1) \tag{3.117}$$

and, since φ_δ is non-decreasing,

$$\sum_{k=k_0+1}^{\infty} \varphi_\delta(c^{-k})c^k(c-1) = \sum_{k=k_0+1}^{\infty} \varphi_\delta(c^{-k}) \int_{c^{-k-1}}^{c^{-k}} t^{-2}\,dt$$
$$\geq \sum_{k=k_0+1}^{\infty} \int_{c^{-k-1}}^{c^{-k}} \varphi_\delta(t)t^{-2}\,dt = \int_0^{c^{-k_0-1}} \varphi_\delta(t)t^{-2}\,dt. \tag{3.118}$$

It follows from (3.116), (3.117), and (3.118) that, for $\delta < \tilde{\delta}(\varepsilon)$,

$$\Lambda_\delta(u_\delta) \geq \frac{1}{c}c_1^{-1}\big[\kappa(c_1\tilde{\varphi}_1) - \varepsilon\big] \int_0^{c^{-k_0-1}} \varphi_\delta(t)t^{-2}\,dt.$$

Note that

$$\lim_{\delta\to 0} \int_0^{c^{-k_0-1}} \varphi_\delta(t)t^{-2}\,dt = \lim_{\delta\to 0} \int_0^{c^{-k_0-1}/\delta} \varphi(t)t^{-2}\,dt$$
$$= \int_0^{\infty} \varphi(t)t^{-2}\,dt = \gamma_d^{-1} \text{ by (1.5)}.$$

On the other hand, by (1.5) applied with $c_1\tilde{\varphi}_1$, we have

$$\gamma_d c_1 \int_1^{\infty} t^{-2}\,dt = \gamma_d c_1 = 1.$$

We derive that

$$\liminf_{\delta\to 0} \Lambda_\delta(u_\delta) \geq \frac{1}{c}(\kappa(c_1\tilde{\varphi}_1) - \varepsilon).$$

Since $c > 1$ and $\varepsilon > 0$ are arbitrary, we obtain (3.108). □

Theorem 5 suggests the following question

Open Problem 4 *Assume that $\varphi,\ \psi \in \mathcal{A}$ satisfy*

$$\varphi \geq \psi \text{ near } 0 \text{ (resp. } \varphi = \psi \text{ near } 0). \tag{3.119}$$

Is it true that

$$K(\varphi) \geq K(\psi) \ \big(\text{resp. } K(\varphi) = K(\psi)\big)?$$

We conclude this section with a simple observation

Proposition 5 *The set $\mathcal{A}$ is convex and the function $\varphi \mapsto \kappa(\varphi)$ is concave on $\mathcal{A}$. Moreover, $t \mapsto \kappa(t\varphi + (1-t)\psi)$ is continuous on $[0, 1]$ for all $\varphi, \psi \in \mathcal{A}$. In particular, $\kappa(\mathcal{A})$ is an interval.*

This is an immediate consequence of the fact that

$$\kappa(\varphi) = \inf \liminf_{\delta \to 0} \Lambda_\delta(v_\delta, \varphi)$$

and that $\varphi \mapsto \Lambda_\delta(v_\delta, \varphi)$ is linear for fixed δ.

3.6 Proof of the Fact that $K(c_1\tilde{\varphi}_1) < 1$ for Every $d \geq 1$

In view of Theorem 1, it suffices to construct a bounded domain $\Omega \subset \mathbb{R}^d$, a function $u \in BV(\Omega)$ with $\int_\Omega |\nabla u| = 1$, a sequence $\delta_n \to 0$, and a sequence $(u_n) \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and

$$\limsup_{n \to +\infty} \Lambda_{\delta_n}(u_n, c_1\tilde{\varphi}_1) < 1. \tag{3.120}$$

We take $\Omega = Q$, $\delta_n = 1/n$, $u(x) = x_1$ where $x = (x_1, x')$ with $x_1 \in (0, 1)$ and $x' \in Q' = (0, 1)^{d-1}$, and

$$u_n(x) = i/n \text{ if } i/n \leq x_1 < (i+1)/n \text{ for } 0 \leq i \leq n-1.$$

Clearly $u_n \to u$ in $L^1(Q)$ as $n \to +\infty$.

It follows from the definition of u_n and u that for $(x, y) \in Q^2$,

$$\text{if } |u_n(x) - u_n(y)| > 1/n, \text{ then } |u(x) - u(y)| > 1/n, \tag{3.121}$$

which implies that

$$\begin{aligned} A_n &:= \big\{(x, y) \in Q^2;\ |u_n(x) - u_n(y)| > 1/n\big\} \subset \\ B_n &:= \big\{(x, y) \in Q^2;\ |u(x) - u(y)| > 1/n\big\}. \end{aligned} \tag{3.122}$$

Thus, by the definition of $\Lambda_{1/n}$,

$$\Lambda_{1/n}(u, c_1\tilde{\varphi}_1) - \Lambda_{1/n}(u_n, c_1\tilde{\varphi}_1) = \frac{c_1}{n} \iint\limits_{B_n \setminus A_n} \frac{1}{|x-y|^{d+1}}\, dx\, dy. \tag{3.123}$$

For $0 \leq i \leq n-2$ and $n \geq 3$, set

$$Z_{i,n} = \big\{(x, y) \in Q^2;\ i/n < x_1 < (i+1/2)/n \text{ and } (i+3/2)/n < y_1 < (i+2)/n\big\}.$$

Note that if $(x, y) \in Z_{i,n}$ we have

$$u(y) - u(x) = y_1 - x_1 > 1/n \quad \text{and} \quad u_n(y) - u_n(x) = (i+1)/n - i/n = 1/n,$$

so that

$$Z_{i,n} \subset B_n \backslash A_n \text{ for } 0 \le i \le n-2.$$

On the other hand if $(x, y) \in Z_{i,n}$ we have

$$|x-y|^2 = |x_1 - y_1|^2 + |x' - y'|^2 \le 4/n^2 + |x' - y'|^2,$$

and consequently

$$\begin{aligned} \iint\limits_{B_n \backslash A_n} \frac{1}{|x-y|^{d+1}}\, dx\, dy &\ge \sum_{i=0}^{n-2} \iint\limits_{Z_{i,n}} \frac{1}{|x-y|^{d+1}}\, dx\, dy \\ &\ge \sum_{i=0}^{n-2} \frac{1}{4n^2} \iint\limits_{Q' \times Q'} \frac{1}{\left((4/n^2) + |x'-y'|^2\right)^{\frac{d+1}{2}}}\, dx' dy' \\ &\sim \frac{1}{n} \iint\limits_{Q' \times Q'} \frac{1}{\left((4/n^2) + |x'-y'|^2\right)^{\frac{d+1}{2}}}\, dx' dy'. \end{aligned} \tag{3.124}$$

Recall the (easy and) standard fact that

$$\iint\limits_{Q' \times Q'} \frac{1}{\left(a^2 + |x'-y'|^2\right)^{\frac{d+1}{2}}}\, dx' dy' \sim 1/a^2 \text{ for small } a. \tag{3.125}$$

Combining (3.123), (3.124), and (3.125) yields

$$\Lambda_{1/n}(u, c_1 \tilde{\varphi}_1) - \Lambda_{1/n}(u_n, c_1 \tilde{\varphi}_1) \ge C_d > 0. \tag{3.126}$$

From Proposition 1, we get

$$\lim_{n \to \infty} \Lambda_{1/n}(u) = \int_Q |\nabla u| = 1. \tag{3.127}$$

The desired result (3.120) follows from (3.126) and (3.127). □

4 Compactness Results. Proof of Theorems 2 and 3

The following subtle estimate from [46, Theorem 1] (with roots in [10]) plays a crucial role in the proof of Theorems 2 and 3.

Lemma 15 *Let $d \geq 1$, B_1 be the unit ball (or cube), and $u \in L^1(B_1)$. There exists a positive constant C_d, depending only on d, such that*

$$\int_{B_1}\int_{B_1} |u(x) - u(y)| \, dx \, dy \leq C_d \Big(\mathop{\int_{B_1}\int_{B_1}}_{|u(x)-u(y)|>1} \frac{1}{|x-y|^{d+1}} \, dx \, dy + 1 \Big). \quad (4.1)$$

By scaling, we obtain, for any ball or cube B,

$$\int_B\int_B |u(x) - u(y)| \, dx \, dy \leq C_d \Big(|B|^{1+1/d} \mathop{\int_B\int_B}_{|u(x)-u(y)|>1} \frac{1}{|x-y|^{d+1}} \, dx \, dy + |B|^2 \Big). \quad (4.2)$$

The reader can find in [16] a connection between these inequalities and the VMO/BMO spaces.

Here is a question related to Lemma 15:

Open Problem 5 *Is it true that*

$$\mathop{\int_{B_1}\int_{B_1}}_{|u(x)-u(y)|>1} |u(x) - u(y)| \, dx \, dy \leq C_d \mathop{\int_{B_1}\int_{B_1}}_{|u(x)-u(y)|>1} \frac{1}{|x-y|^{d+1}} \, dx \, dy \quad \forall u \in L^1(B_1)? \quad (4.3)$$

4.1 Proof of Theorem 2

In this subsection we fix $\delta = 1$. We recall the notation from (1.1)

$$\Lambda(u, \Omega) = \Lambda_1(u, \Omega) = \int_\Omega\int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x-y|^{d+1}} \, dx \, dy.$$

Here is an immediate consequence of Lemma 15.

Lemma 16 *Let B be a ball (or cube) and φ be such that (1.4) and (1.8) hold, and let $u \in L^1(B)$. We have*

$$\int_B\int_B |u(x)-u(y)| \, dx \, dy \leq C_d \Big\{ \frac{\lambda}{\varphi(\lambda)} |B|^{1+1/d} \Lambda(u, B) + \lambda |B|^2 \Big\}, \quad \forall \lambda > 0. \quad (4.4)$$

Assume Ω is bounded. Denote $\Gamma = \partial\Omega$, and set

$$\Omega_t = \{x \in \mathbb{R}^d; \text{ dist }(x, \Omega) < t\}.$$

For t small enough, every $x \in \Omega_t \setminus \Omega$ can be uniquely written as

$$x = x_\Gamma + s\nu(x_\Gamma), \quad (4.5)$$

where x_Γ is the projection of x onto Γ, $s = \text{dist }(x, \Gamma)$, and $\nu(y)$ denotes the outward normal unit vector at $y \in \Gamma$.

Lemma 17 *Assume that Ω is bounded. Fix $t > 0$ small enough such that (4.5) holds for any $x \in \Omega_t$. There exists an extension U of u in Ω_t such that*

$$\|U\|_{L^1(\Omega_t)} \le C\|u\|_{L^1(\Omega)} \quad \textit{and} \quad \Lambda(U, \Omega_t) \le C\Lambda(u, \Omega),$$

for some positive constant C depending only on Ω.

Proof Define

$$U(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ u\big(x_\Gamma - s\nu(x_\Gamma)\big) & \text{if } x \in \Omega_t \setminus \Omega. \end{cases}$$

It is clear that

$$\|U\|_{L^1(\Omega_t)} \le C\|u\|_{L^1(\Omega)}.$$

In this proof C denotes a positive constant depending only on Ω. It remains to prove that

$$\Lambda(U, \Omega_t) \le C\Lambda(u, \Omega).$$

By the definition of Λ in (1.1), it suffices to show that

$$\int_{\Omega_t} dy \int_{\Omega_t \setminus \Omega} \frac{\varphi(|U(x) - U(y)|)}{|x - y|^{d+1}}\, dx \le C \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{d+1}}\, dx\, dy. \tag{4.6}$$

If $x \in \Omega_t \setminus \Omega$ and $y \in \Omega_t \setminus \Omega$, then

$$U\big(x_\Gamma + s_1\nu(x_\Gamma)\big) - U\big(y_\Gamma + s_2\nu(y_\Gamma)\big) = u\big(x_\Gamma - s_1\nu(x_\Gamma)\big) - u\big(y_\Gamma - s_2\nu(y_\Gamma)\big),$$

and

$$\Big|\big(x_\Gamma + s_1\nu(x_\Gamma)\big) - \big(y_\Gamma + s_2\nu(y_\Gamma)\big)\Big| \ge C\Big|\big(x_\Gamma - s_1\nu(x_\Gamma)\big) - \big(y_\Gamma - s_2\nu(y_\Gamma)\big)\Big|,$$

and, if $x \in \Omega_t \setminus \Omega$ and $y \in \Omega$, then

$$U\big(x_\Gamma + s_1\nu(x_\Gamma)\big) - U(y) = u\big(x_\Gamma - s_1\nu(x_\Gamma)\big) - u(y),$$

and

$$\Big|\big(x_\Gamma + s_1\nu(x_\Gamma)\big) - y\Big| \ge C\Big|\big(x_\Gamma - s_1\nu(x_\Gamma)\big) - y\Big|.$$

Hence (4.6) holds. □

We are ready to present the

Proof of Theorem 2 It suffices to consider the case where Ω is bounded. By Lemma 17, one only needs to prove that up to a subsequence, $u_n \to u$ in $L^1_{loc}(\Omega)$. For a cube Q in Ω, define

$$F(u, Q) = \int_Q \int_Q \frac{\varphi\big(|u(x) - u(y)|\big)}{|x - y|^{d+1}}\, dx\, dy + |Q|.$$

Since, by (4.4),

$$\frac{1}{|Q|}\int_Q\int_Q |u(x)-u(y)|\,dx\,dy$$
$$\leq C_d\Big\{\frac{\lambda}{\varphi(\lambda)}|Q|^{1/d}\int_Q\int_Q \frac{\varphi\big(|u(x)-u(y)|\big)}{|x-y|^{d+1}}\,dx\,dy+\lambda|Q|\Big\},$$

it follows that

$$\frac{1}{|Q|}\int_Q\int_Q |u(x)-u(y)|\,dx\,dy \leq \rho(|Q|)F(u,Q), \tag{4.7}$$

where

$$\rho(t):=C_d\inf_{\lambda>0}\Big(\frac{\lambda t^{1/d}}{\varphi(\lambda)}+\lambda\Big).$$

It is clear that ρ is non-decreasing and, by (1.8),

$$\lim_{t\to 0}\rho(t)=0. \tag{4.8}$$

For $\varepsilon>0$ and $n\in\mathbb{N}$, set

$$u_{n,\varepsilon}(x)=\frac{1}{\varepsilon^d}\int_{Q_\varepsilon(x)} u_n(y)\,dy,$$

where $Q_\varepsilon(x)$ is the cube centered at x of side ε. Fix an arbitrary cube $Q\subset\subset\Omega$. We claim that

$$\int_Q |u_n(x)-u_{n,\varepsilon}(x)|\,dx \to 0 \text{ as } \varepsilon\to 0, \text{ uniformly in } n. \tag{4.9}$$

Let ε be small enough such that $Q+\varepsilon[-1,1]^d\subset\Omega$. Then there exists a finite family $\big(Q(j)\big)_{i\in J}$ of disjoint open ε-cubes such that

$$Q\subset \text{ interior }\Big(\bigcup_{j\in J}\overline{Q(j)}\Big)\subset\bigcup_{j\in J}\overline{2Q(j)}\subset\Omega,$$

and thus $\mathrm{card}J\sim 1/\varepsilon^d$. Here and in what follows $aQ(j)$ denotes the cube which has the same center as $Q(j)$ and of a times its length. Applying (4.7) we have

$$\int_{Q(j)}|u_n(x)-u_{n,\varepsilon}(x)|\,dx \leq \frac{C}{|2Q(j)|}\int_{2Q(j)}\int_{2Q(j)}|u_n(x)-u_n(y)|\,dx\,dy$$
$$\leq C\rho(2^d\varepsilon^d)F\big(u_n,2Q(j)\big), \tag{4.10}$$

since $Q(j) + \varepsilon[-1/2, 1/2]^d \subset 2Q(j)$. Note that the family $2Q(j)$ is not disjoint, however, they have a finite number of overlaps (depending only on d). Therefore, for any $f \geq 0$,

$$\sum_j \int_{2Q(j)} \int_{2Q(j)} f \leq C \int_\Omega \int_\Omega f, \tag{4.11}$$

Summing with respect to j in (4.10), we derive from (4.11) that

$$\int_Q |u_n(x) - u_{n,\varepsilon}(x)|\, dx \leq C\rho(2^d \varepsilon^d) F\big(u_n, \Omega\big). \tag{4.12}$$

Using (1.13), (4.8), and (4.12), we obtain (4.9). It follows from (4.9) and a standard argument (see, e.g., the proof of the theorem of Riesz-Frechet-Kolmogorov in [13, Theorem 4.26]) that there exists a subsequence (u_{n_k}) of (u_n) and $u \in L^1_{loc}(\Omega)$ such that (u_{n_k}) converges to u in $L^1_{loc}(\Omega)$. $\square$

Remark 8 Using Theorem 2, one can prove that Λ is lower semi-continuous with respect to weak convergence in L^q for any $q \geq 1$.

4.2 Proof of Theorem 3

It suffices to consider the case where Ω is bounded. By Lemma 17, one only needs to prove that up to a subsequence, $u_n \to u$ in $L^1_{loc}(\Omega)$. Fix $\lambda_0 > 0$ such that $\varphi(\lambda_0) > 0$. Without loss of generality, one may assume that $\lambda_0 = 1$. From (1.14), we have

$$\sup_n \int_\Omega \int_\Omega_{|u_n(x)-u_n(y)|>\delta_n} \frac{\delta_n}{|x-y|^{d+1}}\, dx\, dy \leq C.$$

We now follow the same strategy as in the proof of Theorem 2. Define

$$u_{n,\varepsilon}(x) = \frac{1}{\varepsilon^d} \int_{Q_\varepsilon(x)} u_n(y)\, dy, \tag{4.13}$$

Here $Q_\varepsilon(x)$ is the cube centered at x of side ε. Fix an arbitrary cube $Q \subset\subset \Omega$. We claim that

$$\lim_{\varepsilon \to 0} \limsup_n \int_Q |u_n(x) - u_{n,\varepsilon}(x)|\, dx = 0. \tag{4.14}$$

Let ε be small enough such that $Q + \varepsilon[-1, 1]^d \subset \Omega$. Then there exists a finite family $\big(Q(j)\big)_{i \in J}$ of disjoint open ε-cubes such that

$$Q \subset \text{ interior}\Big(\bigcup_{j \in J} \overline{Q(j)}\Big) \subset \bigcup_{j \in J} \overline{2Q(j)} \subset \Omega.$$

We have

$$\int_{Q(j)} |u_n(x) - u_{n,\varepsilon}(x)|\, dx \le \frac{C}{|2Q(j)|} \int_{2Q(j)} \int_{2Q(j)} |u_n(x) - u_n(y)|\, dx\, dy, \quad (4.15)$$

since $Q(j) + \varepsilon[-1/2, 1/2]^d \subset 2Q_j$. By (4.15) and (4.2) with $B = 2Q(j)$, we have

$$\int_{Q(j)} |u_n(x) - u_{n,\varepsilon}(x)|\, dx \le \kappa \Big(\varepsilon \int_{2Q(j)} \int_{\substack{2Q(j) \\ |u_n(x)-u_n(y)|>\delta_n}} \frac{\delta_n}{|x-y|^{d+1}}\, dx\, dy + \delta_n \varepsilon^d \Big). \quad (4.16)$$

Summing with respect to j in (4.16), we obtain

$$\int_Q |u_n(x) - u_{n,\varepsilon}(x)|\, dx \le C(\varepsilon + \delta_n).$$

Clearly, for fixed n,

$$\lim_{\varepsilon \to 0} \int_Q |u_n(x) - u_{n,\varepsilon}(x)|\, dx = 0.$$

Therefore (4.14) holds and we conclude as in the proof of Theorem 2. □

5 Some Functionals Related to Image Processing

Given $q \ge 1$, $\lambda > 0$, $\delta > 0$, $d \ge 1$, Ω a smooth bounded open subset of $\mathbb{R}^d$, and $f \in L^q(\Omega)$, consider the non-local, non-convex functional defined on $L^q(\Omega)$ by

$$E_\delta(u) = \lambda \int_\Omega |u - f|^q + \Lambda_\delta(u) := \lambda \int_\Omega |u - f|^q + \delta \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|/\delta)}{|x-y|^{d+1}}\, dx\, dy. \quad (5.1)$$

Our goal in this section is twofold. In the first part, we investigate the existence of a minimizer for E_δ (δ is fixed) and then we study the behavior of these minimizers (or almost minimizers) as $\delta \to 0$. In the second part, we explain how these results are connected to Image Processing.

5.1 Variational Problems Associated with E_δ

We start with an immediate consequence of Theorem 2.

Corollary 7 *Let $\delta > 0$ be fixed. Assume that $\varphi \in \mathcal{A}$ satisfies (1.8). There exists $u \in L^q(\Omega)$ such that*

$$E_\delta(u) = \inf_{w \in L^q(\Omega)} E_\delta(w).$$

As we know from Theorem 1, under assumptions (1.2)-(1.5), (Λ_δ) Γ-converges to $K \int_\Omega |\nabla \cdot|$ as $\delta \to 0$, for some constant $0 < K \le 1$. Therefore, one may expect that the minimizers of E_δ converge to the unique minimizer in $L^q(\Omega)$ of E_0, where

$$E_0(w) = \lambda \int_\Omega |w - f|^q + K \int_\Omega |\nabla w|. \tag{5.2}$$

If one does not assume (1.8) one can not apply Corollary 7, and minimizers of E_δ might not exist; however, one can always consider almost minimizers. Here is slight generalization of Theorem 4.

Theorem 6 *Let $d \geq 1$, $q \geq 1$, Ω be a smooth bounded open subset of $\mathbb{R}^d$, $f \in L^q(\Omega)$, and $\varphi \in \mathcal{A}$. Let (δ_n), (τ_n) be two positive sequences converging to 0 as $n \to \infty$ and $u_n \in L^q(\Omega)$ be such that*

$$E_{\delta_n}(u_n) \leq \inf_{u \in L^q(\Omega)} E_{\delta_n}(u) + \tau_n. \tag{5.3}$$

Then $u_n \to u_0$ in $L^q(\Omega)$ where u_0 is the unique minimizer of the functional E_0 defined on $L^q(\Omega) \cap BV(\Omega)$ by

$$E_0(u) := \lambda \int_\Omega |u - f|^q + K \int_\Omega |\nabla u|,$$

and $0 < K \leq 1$ is the constant in Theorem 1.

Proof It is clear that (u_n) is bounded in $L^1(\Omega)$. By Theorem 3, there exists a subsequence (u_{n_k}) which converges to some u_0 a.e. and in $L^1(\Omega)$. It follows from Fatou's lemma and Property (G1) in Sect. 3 that

$$E_0(u_0) \leq \liminf_{k \to \infty} E_{\delta_{n_k}}(u_{n_k}). \tag{5.4}$$

We will prove that u_0 is the unique minimizer of E_0 in $L^q(\Omega) \cap BV(\Omega)$. Let $v \in L^q(\Omega) \cap BV(\Omega)$ be the unique minimizer of E_0. Applying Theorem 1, there exists $v_n \in L^1(\Omega)$ such that $v_n \to v$ in L^1 (without loss of generality, one may assume that $v_n \to v$ a.e.) and

$$\limsup_{n \to \infty} \Lambda_{\delta_n}(v_n) \leq K \int_\Omega |\nabla v|.$$

For $A > 0$, recall the notation T_A defined in (2.39). From (1.4), we have

$$\Lambda_{\delta_n}(T_A v_n) \leq \Lambda_{\delta_n}(v_n).$$

By definition of u_n, we obtain

$$E_{\delta_n}(u_n) \leq \lambda \int_\Omega |T_A v_n - f|^q + \Lambda_{\delta_n}(T_A v_n) + \tau_n \leq \lambda \int_\Omega |T_A v_n - f|^q + \Lambda_{\delta_n}(v_n) + \tau_n.$$

Letting $n \to \infty$ yields

$$E_0(u_0) \leq \liminf_{n \to \infty} E_{\delta_n}(u_n) \leq \lambda \int_\Omega |T_A v - f|^q + K \int_\Omega |\nabla v|.$$

As $A \to \infty$, we find

$$E_0(u_0) \le \lambda \int_\Omega |v - f|^q + K \int_\Omega |\nabla v|. \tag{5.5}$$

This implies that u_0 is the unique minimizer of E_0.
We next prove that $u_n \to u_0$ in L^q. Since

$$E_0(u_0) \ge \limsup_{n\to\infty} E_{\delta_n}(u_n) \ge E_0(u_0)$$

by (5.5), and

$$\liminf_{n\to\infty} \Lambda_{\delta_n}(u_n) \ge K \int_\Omega |\nabla u_0|,$$

by Theorem 1, we have

$$\lim_{n\to\infty} \int_\Omega |u_n - f|^q = \int_\Omega |u_0 - f|^q.$$

In addition we know that $u_n - f \to u_0 - f$ a.e. in Ω. Therefore $u_n - f \to u_0 - f$ in $L^q(\Omega)$; thus $u_n \to u_0$ in $L^q(\Omega)$. The proof is complete. □

Remark 9 In case a Lavrentiev - type gap does occur (see Open problem 3 and the subsequent comments) it would be interesting to investigate what happens in Theorem 6 if $E_{\delta|L^q(\Omega)}$ is replaced by $E_{\delta|C^0(\bar{\Omega})}$ (with numerous possible variants).

5.2 Connections with Image Processing

A fundamental challenge in Image Processing is to improve images of poor quality. Denoising is an immense subject, see, e.g., the excellent survey by A. Buades, B. Coll and J. M. Morel [21]. One possible strategy is to introduce a filter F and use a variational formulation

$$\min_u \Big\{\lambda \int_\Omega |u - f|^2 + F(u)\Big\}, \tag{5.6}$$

or, alternatively, the associated Euler equation

$$2\lambda(u - f) + F'(u) = 0. \tag{5.7}$$

Here f is the given image of poor quality, $\lambda > 0$ is the fidelity parameter (fixed by experts) which governs how much filtering is desirable. Minimizers of (5.6) (or solutions to (5.7)) are the denoised images.

Many types of filters are used in Image Processing. Here are three popular ones. The first one is the celebrated (ROF) filter due L. Rudin, S. Osher, and E. Fatemi [52]:

$$F(u) = \int_\Omega |\nabla u|$$

(see also [28,29,37]). The corresponding minimization problem is

$$(ROF) \qquad \min_{u\in L^2(\Omega)} \Big\{ \lambda \int_\Omega |u-f|^2 + \int_\Omega |\nabla u| \Big\}.$$

The functional in (ROF) is strictly convex. It follows from standard Functional Analysis that, given $f \in L^2(\Omega)$, there exists a unique minimizer $u_0 \in BV(\Omega) \cap L^2(\Omega)$.

The second filter, due to G. Gilboa and S. Osher [35] (see also [36]), is

$$F(u) = \int_\Omega \Big(\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^2} w(x,y)\, dy \Big)^{1/2} dx,$$

where w is a given weight function. The corresponding minimization problem is

$$(GO) \qquad \min_{u\in L^2(\Omega)} \Big\{ \lambda \int_\Omega |u-f|^2 + \int_\Omega \Big(\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^2} w(x,y)\, dy \Big)^{1/2} dx \Big\}.$$

The functional in (GO) is strictly convex. Again by standard Functional Analysis, there exists a unique minimizer u_0 of (GO). One can prove (see [17]) that if $w(x,y) = \rho_\varepsilon(|x-y|)$, where (ρ_ε) is a sequence of mollifiers as in Remark 4, then the corresponding minimizers (u_ε) of (GO_ε) (i.e., (GO) with $w(x,y) = \rho_\varepsilon(|x-y|)$) converge, as $\varepsilon \to 0$, to the unique solution of the (ROF_k) problem

$$(ROF_k) \qquad \min_{u\in L^2(\Omega)\cap BV(\Omega)} \Big\{ \lambda \int_\Omega |u-f|^2 + k \int_\Omega |\nabla u| \Big\},$$

where

$$k = \Big(\int_{\mathbb{S}^{d-1}} |\sigma \cdot e|^2 \, d\sigma \Big)^{1/2},$$

for some $e \in \mathbb{S}^{d-1}$. The proof in [17] is strongly inspired by the results of J. Bourgain, H. Brezis, and P. Mironescu [7], A. Ponce [49], and G. Leoni and D. Spector [41].

In a similar spirit, G. Aubert and P. Kornprobst in [6] have proposed to use the filter

$$F(u) = I_\varepsilon(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_\varepsilon(|x-y|)\, dx\, dy,$$

and the corresponding minimization problem is

$$(AK_\varepsilon) \qquad \min_{u\in L^2} \Big\{ \lambda \int_\Omega |u-f|^2 + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_\varepsilon(|x-y|)\, dx\, dy \Big\}.$$

As above, (AK_ε) admits a unique minimizer u_ε and, as $\varepsilon \to 0$, (u_ε) converges to the solution of (ROF_{γ_d}) where γ_d is the constant defined in (1.5).

More recently, J.-F. Cai, B. Dong, S. Osher, and Z. Shen [27] have studied a general version of (ROF) of the type

$$\inf_u \left\{ \nu \|Du\|_* + \frac{1}{2} \int_\Omega |Au - f|^2 \right\},$$

where ν is a positive constant, D is a linear differential operator, A is a bounded linear operator, and $\| \ \|_*$ is a properly chosen norm. They introduce a discretized version

$$E_n(u) = \nu \|D_n u\|_* + \frac{1}{2} \int_\Omega |A_n u - f_n|^2$$

of

$$E(u) = \nu \|Du\|_* + \frac{1}{2} \int_\Omega |Au - f|^2$$

and prove that (E_n) converges to E **both** pointwise and in the sense of Γ-convergence. This is again a situation where the pointwise limit and the Γ-limit **coincide**. Consequently, (almost) minimizers of E_n converge to (almost) minimizers of E.

The third type of filter was introduced in the pioneering works of L. S. Lee [39] and L. P. Yaroslavsky (see [55,56]); more details can be found in the expository paper by A. Buades, B. Coll, and J. M. Morel [21]; see also [22,23,48,53]) where the terms "neighbourhood filters", "non-local means" and "bilateral filters" are used. Originally, they were not formulated as variational problems. In an important paper K. Kindermann, S. Osher and P. W. Jones [38] showed that some of these filters come from the Euler-Lagrange equation of a minimization problem where the functional F has the form

$$F(u) = \int_\Omega \int_\Omega \varphi\big(|u(x) - u(y)|/\delta\big) w(|x - y|)\, dx\, dy,$$

$\delta > 0$ is a fixed small parameter, φ is a given **non-convex** function, and $w \geq 0$ is a weight function. The corresponding minimization problem is

$$(YNF_\delta) \qquad \min_{u \in L^2} \left\{ \lambda \int_\Omega |u - f|^2 + \int_\Omega \int_\Omega \varphi\big(|u(x) - u(y)|/\delta\big) w(|x - y|)\, dx\, dy \right\}.$$

Here are some examples of $\varphi's$ and $w's$ used in Image Processing see, e.g., [38, Section 3]:

(i) $\varphi = \tilde{\varphi}_2$ or $\varphi = \tilde{\varphi}_3$ (from the list of examples in the Introduction).
(ii) $w = 1$ or

$$w(t) = \begin{cases} 1 & \text{if } t < \rho, \\ 0 & \text{otherwise,} \end{cases}$$

for some $\rho > 0$.

In this paper, we suggest a new example for w:

$$w(|x-y|) = \frac{1}{|x-y|^{d+1}}. \tag{5.8}$$

Taking $\lambda \sim 1/\delta$, more precisely $\lambda = \gamma/\delta$, we are led to the minimization problem:

$$\min_{u \in L^2} \left\{ \gamma \int_\Omega |u-f|^2 + \Lambda_\delta(u) \right\}. \tag{5.9}$$

Up to now, there was **no** rigorous analysis whatsoever for problems of the form (YNF_δ). Even the existence of minimizers in (YNF_δ), for fixed δ, was lacking. Our contributions for the new choice of w in (5.8) are twofold:

1. Existence of minimizers for (5.9) under fairly general assumptions on φ (Theorem 2).
2. Asymptotic analysis as $\delta \to 0$: $(YNF_\delta) \to (ROF_K)$ (Theorem 4).

Acknowledgements We are extremely grateful to J. Bourgain for sharing fruitful ideas which led to the joint work [10] with the second author. Some of the techniques developed in [10] served as a source of inspiration for many subsequent works. The first author (H.B.) warmly thanks R. Kimmel and J. M. Morel for useful discussions concerning Image Processing.

A Appendix: Proof of Pathology 2

We construct a function $u \in W^{1,1}(0,1)$ such that, for $\varphi = c_1 \tilde{\varphi}_1$,

$$\liminf_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u| \quad \text{and} \quad \limsup_{\delta \to 0} \Lambda_\delta(u) = +\infty. \tag{A1}$$

Set $x_n = 1 - 1/n$ for $n \geq 1$. Set $\delta_1 = 1/100$ and $T_1 = e^{-\delta_1^{-1}}$. Let y_1 be the middle point of the interval $(x_1, x_1 + T_1)$ and fix $0 < t_1 < T_1/4$ such that

$$\int_{x_1}^{y_1 - t_1} dx \int_{y_1 + t_1}^{x_1 + T_1} \frac{\delta_1}{|x-y|^2}\, dy \geq 1.$$

Since

$$\int_\alpha^\beta dx \int_\beta^\gamma \frac{\delta_1}{|x-y|^2}\, dy = +\infty, \tag{A2}$$

for all $\alpha < \beta < \gamma$, such a t_1 exists. Define $u_1 \in W^{1,1}(0,1)$ by

$$u_1(x) = \begin{cases} \text{constant in} & [0, x_1], \\ \text{affine in} & [x_1, y_1 - t_1], \\ \text{affine in} & [y_1 - t_1, y_1 + t_1], \\ \text{affine in} & [y_1 + t_1, x_1 + T_1], \\ \text{constant in} & [x_1 + T_1, 1], \end{cases} \quad \text{and} \quad \begin{cases} u_1(x_1) = 0, \\ u_1(y_1 - t_1) = \delta_1/3, \\ u_1(y_1 + t_1) = 2\delta_1/3, \\ u_1(x_1 + T_1) = \delta_1. \end{cases}$$

Assuming that δ_k, T_k, t_k, and u_k are constructed for $1 \le k \le n-1$ and for $n \ge 2$ such that u_k is Lipschitz. We then obtain δ_n, T_n, t_n, and u_n as follows. Fix $0 < \delta_n < \delta_{n-1}/8$ sufficiently small such that

$$\Lambda_{2\delta_n}(u_{n-1}) + \int_0^{x_{n-1}+T_{n-1}} dx \int_{x_n}^1 \frac{2\delta_n c_1}{|x-y|^2}\, dy \le \int_0^1 |u_{n-1}'| + 1/n. \tag{A3}$$

Such a constant δ_n exists by Proposition 1 (in fact u_{n-1} is only Lipschitz; however Proposition 1 holds as well for Lipschitz functions, see also Proposition C1). Set $T_n = e^{-\delta_n^{-1}}$ and let y_n be the middle point of the interval $(x_n, x_n + T_n)$ and fix $0 < t_n < T_n/4$ such that

$$\int_{x_n}^{y_n - t_n} dx \int_{y_n+t_n}^{x_n+T_n} \frac{\delta_n}{|x-y|^2}\, dy \ge n. \tag{A4}$$

Such a t_n exists by (A2). Define a continuous function $w_n : [0,1] \mapsto [0,1]$, $n \ge 2$, as follows

$$w_n(x) = \begin{cases} \text{constant in } & [0, x_n], \\ \text{affine in } & [x_n, y_n - t_n], \\ \text{affine in } & [y_n - t_n, y_n + t_n], \\ \text{affine in } & [y_n + t_n, x_n + T_n], \\ \text{constant in } & [x_n + T_n, 1], \end{cases} \quad \text{and} \quad \begin{cases} w_n(0) = 0, \\ w_n(y_n - t_n) = \delta_n/3, \\ w_n(y_n + t_n) = 2\delta_n/3, \\ w_n(x_n + T_n) = \delta_n. \end{cases}$$

Set

$$u_n = u_{n-1} + w_n \text{ in } (0,1).$$

Since w_n and u_{n-1} are Lipschitz, it follows that u_n is Lipschitz. Moreover, one can verify that (u_n) converges in $W^{1,1}(0,1)$ by noting that

$$\|w_n\|_{W^{1,1}(0,1)} \le 2\delta_n \le 2\delta_1/8^{n-1}.$$

Let u be the limit of (u_n) in $W^{1,1}(0,1)$. We derive from the construction of u_n that u is non-decreasing, and for $n \ge 1$,

$$u(x) = u_n(x) \text{ for } x \le x_{n+1}, \tag{A5}$$

$$u \text{ is constant in } (x_n + T_n, x_{n+1}), \tag{A6}$$

$$u(1) - u(x_n) \le \sum_{k \ge n} \delta_k < 2\delta_n, \tag{A7}$$

since $\delta_k < \delta_{k-1}/8$. We have

$$
\begin{aligned}
\Lambda_{2\delta_n}(u) &= \int_0^1 \int_0^1 \frac{\varphi_{2\delta_n}(|u(x)-u(y)|)}{|x-y|^2}\,dx\,dy \\
&= \int_0^{x_n}\int_0^{x_n} \cdots + \int_{x_n}^1\int_{x_n}^1 \cdots + 2\int_0^{x_n}\int_{x_n}^1 \cdots \quad \text{where } \cdots \\
&= \frac{\varphi_{2\delta_n}(|u(x)-u(y)|)}{|x-y|^2}.
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
\int_0^{x_n}\int_0^{x_n} \cdots &\overset{\text{by (A5)}}{\le} \Lambda_{2\delta_n}(u_{n-1}), \\
\int_{x_n}^1\int_{x_n}^1 \cdots &\overset{\text{by (A7)}}{=} 0,
\end{aligned}
$$

and

$$
2\int_0^{x_n}\int_{x_n}^1 \cdots \overset{\text{by (A7)}}{\le} \int_0^{x_{n-1}+T_{n-1}} dx \int_{x_n}^1 \frac{2\delta_n c_1}{|x-y|^2}\,dy,
$$

since u is constant in $[x_{n-1}+T_{n-1}, x_n]$. It follows from (A3) that

$$
\Lambda_{2\delta_n}(u) \le \int_0^1 |u_{n-1}'| + 1/n. \tag{A8}
$$

On the other hand, from (A4), (A5), and the definition of w_n, we have, for $n \ge 1$,

$$
\begin{aligned}
\Lambda_{\delta_n/3}(u) &\ge \int_{x_n}^{y_n-t_n} dx \int_{y_n+t_n}^{x_n+T_n} \frac{\varphi_{\delta_n/3}(|u(x)-u(y)|)}{|x-y|^2}\,dy \\
&= \int_{x_n}^{y_n-t_n} dx \int_{y_n+t_n}^{x_n+T_n} \frac{\varphi_{\delta_n/3}(|w_n(x)-w_n(y)|)}{|x-y|^2}\,dy \\
&\ge \int_{x_n}^{y_n-t_n} dx \int_{y_n+t_n}^{x_n+T_n} \frac{c_1\delta_n/3}{|x-y|^2}\,dy \ge c_1 n/3.
\end{aligned} \tag{A9}
$$

Combining (A8) and (A9) and noting that $u_n \to u$ in $W^{1,1}(0,1)$, we obtain the conclusion. □

B Appendix: Proof of Pathology 3

We first establish (2.41) for $\varphi = c_1\tilde{\varphi}_1$ where $c_1 = 1/2$ is the normalization constant.

Let $c \geq 5$ and for each $k \in \mathbb{N}$ ($k \geq 4$) define a non-decreasing continuous function $v_k : [0, 1] \mapsto [0, 1]$ with $v_k(1) = 1$ as follows

$$v_k(x) = \begin{cases} i/k & \text{for } x \in [i/k, (i+1)/k - 1/(ck)] \quad \forall\, i = 0, \cdots, k-1, \\ \text{affine} & \text{for } x \in [(i+1)/k - 1/(ck), (i+1)/k] \quad \forall\, i = 0, \cdots, k-1. \end{cases} \tag{B1}$$

Clearly,

$$\text{if } |v_k(x) - v_k(y)| > 1/k \text{ then } |x - y| > 1/k. \tag{B2}$$

Define

$$V_k(x) := \lim_{c \to +\infty} v_k(x) \quad \text{for } x \in [0, 1].$$

Since $c_1 = 1/2$, one can show that (see [43, page 683])

$$\begin{aligned} A_0 := \limsup_{k \to \infty} \Lambda_{1/k}(V_k) = \limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-2} \int_{i/k}^{(i+1)/k} dx \\ \times \int_{(i+2)/k}^{1} \frac{1}{|x-y|^2}\, dy < 1 = \lim_{\delta \to 0} \Lambda_\delta(x, [0, 1]). \end{aligned}$$

Since, for $c \geq 2$,

$$\begin{aligned} &\frac{1}{k} \sum_{i=0}^{k-2} \int_{i/k}^{(i+1)/k} dx \int_{(i+2)/k - 1/(ck)}^{(i+2)/k} \frac{1}{|x-y|^2}\, dy \\ &\quad \leq \frac{k-1}{k} \frac{1}{k} \frac{1}{ck} \Big(\frac{1}{k} - \frac{1}{ck}\Big)^{-2} \leq \frac{1}{c}\Big(1 - \frac{1}{c}\Big)^{-2} \leq \frac{4}{c}, \end{aligned}$$

it follows that, for sufficiently large c,

$$\limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-2} \int_{i/k}^{(i+1)/k} dx \int_{(i+2)/k - 1/(ck)}^{1} \frac{1}{|x-y|^2}\, dy \leq \frac{A_0 + 1}{2} < 1. \tag{B3}$$

Fix such a constant c. We are now going to define by induction a sequence of u_n : $[0, 1] \mapsto [0, 1]$. Set

$$u_0 = v_4.$$

Assume that u_{n-1} ($n \geq 1$) is defined and satisfies the following properties:

$$u_{n-1} \text{ is non-decreasing, continuous, and piecewise affine,} \quad u_{n-1}(0) = 0, \tag{B4}$$

and there exists a partition $0 = t_{0,n-1} < t_{1,n-1} < \cdots < t_{2l_{n-1},n-1} = 1$ such that, with the notation $J_{i,n-1} = [t_{i,n-1}, t_{i+1,n-1}]$, the following four properties hold:

$$u_{n-1} \text{ is constant on } J_{2i,n-1} \quad \text{for } i = 0, \cdots, l_{n-1} - 1, \tag{B5}$$

$$u_{n-1} \text{ is affine and not constant on } J_{2i+1,n-1} \quad \text{for } i = 0, \cdots, l_{n-1} - 1, \tag{B6}$$

the total variation of u_{n-1} on the interval $J_{i,n-1}$ with i odd (where u_{n-1} is not constant) is always $1/l_{n-1}$, i.e.,

$$u_{n-1}(t_{2i+2,n-1}) - u_{n-1}(t_{2i+1,n-1}) = 1/l_{n-1} \quad \text{for } i = 0, \cdots, l_{n-1} - 1, \tag{B7}$$

and the intervals $J_{i,n-1}$ with i odd have the same length which is less than the one of any interval $J_{i,n-1}$ with i even, i.e.,

$$|J_{1,n-1}| = |J_{3,n-1}| = \cdots = |J_{2l_{n-1}-1,n-1}| < |J_{2i,n-1}| \quad \text{for } i = 0, \cdots, l_{n-1} - 1. \tag{B8}$$

Since $u_{n-1}(0) = 0$, it follows from the properties of u_{n-1} in (B5) and (B6) that

$$u_{n-1}(t) = s/l_{n-1} + i/l_{n-1} \text{ for } t \in J_{2i+1,n-1} \text{ where } s = (t - t_{2i+1,n-1})/|J_{2i+1,n-1}|. \tag{B9}$$

Set

$$B_{n-1} = \bigcup_{i=0}^{l_{n-1}-1} J_{2i,n-1} \tag{B10}$$

(B_{n-1} is the union of all intervals on which u_{n-1} is constant). For $n \in \mathbb{N}$, let k_n be a sufficient large integer such that

$$\frac{1}{k_n}\int_{-1}^{0} dx \int_{\tau_n}^{1} \frac{1}{|x-y|^2}\, dy < \frac{1}{n} \quad \text{where} \quad \tau_n = |J_{1,n-1}|/k_n \tag{B11}$$

and

$$\frac{1}{k_n}\sum_{i=0}^{k_n-2}\int_{i/k_n}^{(i+1)/k_n} dx \int_{(i+2)/k_n - 1/(ck_n)}^{1} \frac{1}{|x-y|^2}\, dy \le \frac{A_0+1}{2}. \tag{B12}$$

Since, for a small positive number τ,

$$\int_{-1}^{0} dx \int_{\tau}^{1} \frac{1}{|x-y|^2}\, dy \le |\ln \tau|,$$

such a constant k_n exists by (B3). Define

$$u_n(t) = \begin{cases} u_{n-1}(t) & \text{if } t \in B_{n-1}, \\ \dfrac{1}{l_{n-1}} v_{k_n}(s) + \dfrac{i}{l_{n-1}} & \text{if } t \in J_{2i+1,n-1} \text{ for some } 0 \le i \le l_{n-1} - 1, \end{cases} \tag{B13}$$

where $s = (t - t_{2i+1,n-1})/|J_{2i+1,n-1}|$. Then u_n satisfies (B4)-(B8) for some l_n and $t_{i,n}$. Since $0 \le v_k(x) \le x$ for $x \in [0, 1]$, we deduce from (B9) and the definition of u_n that $u_n \le u_{n-1}$. On the other hand, we derive from (B9) and (B13) that, for $m \ge n$,

$$\|u_m - u_n\|_{L^\infty(0,1)} \le 1/l_n.$$

Hence the sequence (u_n) is Cauchy in $C([0, 1])$. Let u be the limit and set

$$\delta_n = 1/(l_{n-1}k_n). \tag{B14}$$

It follows from the definition of u_n and u that

$$u(t) = u_n(t) \text{ for } t = t_{i,n-1} \text{ with } 0 \le i \le 2l_{n-1}. \tag{B15}$$

From the construction of u_n in (B13), the property of v_k in (B2), and (B8), we derive that

$$\text{if } |u(x) - u(y)| > \delta_n, \text{ then } |x - y| > \tau_n, \tag{B16}$$

where τ_n is defined in (B11). Since u_{n-1} is constant in $J_{2i,n-1}$ for $0 \le i \le l_{n-1} - 1$ by (B5), it follows from (B13) that u is constant in $J_{2i,n-1}$ for $0 \le i \le l_{n-1} - 1$. We derive that

$$\begin{aligned}\int_0^1 \int_0^1_{|u(x)-u(y)|>\delta_n} \frac{\delta_n}{|x-y|^2}\,dx\,dy &\le \sum_{i=0}^{l_{n-1}-1} \iint_{\substack{J^2_{2i+1,n-1}\\ |u(x)-u(y)|>\delta_n}} \frac{\delta_n}{|x-y|^2}dx\,dy \\ &+ \sum_{i=0}^{2l_{n-1}-1} \int_{J_{i,n-1}} dx \int_{[0,1]\setminus J_{i,n-1}} \frac{\delta_n}{|x-y|^2}\,dy.\end{aligned} \tag{B17}$$

(with the constraint $|u(x)-u(y)|>\delta_n$ under the inner integrals of the second sum)

Using (B16), we have, by (B11),

$$\sum_{i=0}^{2l_{n-1}-1} \int_{J_{i,n-1}} dx \int_{\substack{[0,1]\setminus J_{i,n-1}\\ |u(x)-u(y)|>\delta_n}} \frac{\delta_n}{|x-y|^2}\,dy \le 4l_{n-1} \int_{-1}^0 dx \int_{\tau_n}^1 \frac{\delta_n}{|x-y|^2}\,dy \le 4/n. \tag{B18}$$

We now estimate, for $0 \le i \le l_{n-1} - 1$,

$$\iint_{\substack{J^2_{2i+1,n-1}\\ |u(x)-u(y)|>\delta_n}} \frac{\delta_n}{|x-y|^2}dx\,dy.$$

Define $g_i : J_{2i+1,n-1} \to [0, 1]$, for $0 \le i \le l_{n-1} - 1$, as follows

$$g_i(x) = (x - t_{2i+1,n-1})/|J_{2i+1,n-1}| \quad \text{for } x \in J_{2i+1,n-1}.$$

We claim that, for $0 \le i \le l_{n-1} - 1$,

$$\text{if } (x, y) \in J^2_{2i+1,n-1}, \quad |u(x) - u(y)| > \delta_n, \quad g_i(x) \in [i/k_n, (i + 1)/k_n], \quad \text{and} \quad x < y,$$

then

$$g_i(y) \in \big[(i + 2)/k_n - 1/(ck_n), 1\big].$$

In fact, if $g_i(z) \in [i/k_n, (i+2)/k_n - 1/(ck_n)]$ then

$$u_n\left(g_i^{-1}\Big(\frac{i}{k_n}\Big)\right) = u\left(g_i^{-1}\Big(\frac{i}{k_n}\Big)\right) \le u(z) \le u\left(g_i^{-1}\Big(\frac{i+2}{k_n} - \frac{1}{ck_n}\Big)\right) = u\left(g_i^{-1}\Big(\frac{i+2}{k_n} - \frac{1}{ck_n}\Big)\right).$$

Here we used (B15) and the fact that u is non-decreasing. It follows from the definition of u_n that, if $g_i(x), g_i(y) \in [i/k_n, (i+2)/k_n - 1/(ck_n)]$ then

$$|u(y) - u(x)| \le \frac{1}{l_{n-1}}\left|v_{k_n}\Big(\frac{i+2}{k_n} - \frac{1}{ck_n}\Big) - v_{k_n}\Big(\frac{i}{k_n}\Big)\right| \le \frac{1}{k_n l_{n-1}} = \delta_n.$$

The claim is proved.

By a change of variables, for $i = 0, \cdots, 2l_{n-1} - 1$,

$$(x, y) \mapsto \Big(g_i(x), g_i(y)\Big) \text{ for } (x, y) \in J^2_{2i+1,n-1},$$

we deduce from the claim that

$$\sum_{i=0}^{l_{n-1}-1} \iint_{\substack{J^2_{2i+1,n-1} \\ |u(x)-u(y)|>\delta_n}} \frac{\delta_n}{|x-y|^2} dx\, dy \le 2l_{n-1}\delta_n \sum_{j=0}^{k_n-2} \int_{j/k_n}^{(j+1)/k_n} dx \int_{(j+2)/k_n - 1/(ck_n)}^{1} \frac{1}{|x-y|^2}\, dy.$$

It follows from (B12) and (B14) that

$$\sum_{i=0}^{l_{n-1}-1} \iint_{\substack{J^2_{2i+1,n-1} \\ |u(x)-u(y)|>\delta_n}} \frac{\delta_n}{|x-y|^2} dx\, dy \le A_0 + 1. \tag{B19}$$

Combining (B17), (B18), and (B19) yields

$$\limsup_{n\to\infty} \int_0^1 \int_0^1_{|u(x)-u(y)|>\delta_n} \frac{\delta_n}{|x-y|^2}\, dx\, dy \le A_0 + 1.$$

Since $c_1 = 1/2$, we have, for $\varphi = c_1\tilde{\varphi}_1$,

$$\limsup_{n\to\infty} \Lambda_{\delta_n}(u) \le (A_0 + 1)/2 < 1. \tag{B20}$$

Note that $u \in C([0,1])$ is non-decreasing and $u(0) = 0$ and $u(1) = 1$. This implies

$$\int_0^1 |u'| = 1.$$

Therefore (2.41) holds for $\varphi = c_1 \tilde{\varphi}_1$ and u.

We next construct a continuous function φ_ℓ which is "close" to $c_1 \tilde{\varphi}_1$ such that (2.41) holds for φ_ℓ and the function u constructed above. For $\ell \ge 1$, define a continuous function $\varphi_\ell : [0, +\infty) \mapsto \mathbb{R}$ by

$$\varphi_\ell(t) = \begin{cases} \alpha_\ell & \text{if } t \ge 1 + 1/\ell, \\ 0 & \text{if } t \le 1, \\ \text{affine} & \text{if } t \in [1, 1 + 1/\ell], \end{cases}$$

where α_ℓ is the constant such that

$$\gamma_1 \int_0^\infty \varphi_\ell(t) t^{-2}\, dt = 2 \int_0^\infty \varphi_\ell(t) t^{-2}\, dt = 1.$$

Then $\varphi_\ell \in \mathcal{A}$. Moreover, $\varphi_\ell(t) \le \alpha_\ell \tilde{\varphi}_1(\beta_\ell t)$ where $\beta_\ell = 1 + 1/\ell$. It follows from (B20) that

$$\liminf_{\delta \to 0} \int_0^1 \int_0^1 \frac{\beta_\ell \delta \varphi_\ell(|u(x) - u(y)|/(\beta_\ell \delta))}{|x-y|^2}\, dx\, dy$$
$$\le \liminf_{\delta \to 0} \int_0^1 \int_0^1 \frac{\alpha_\ell \beta_\ell \delta \tilde{\varphi}_1\Big(|u(x) - u(y)|/\delta\Big)}{|x-y|^2}\, dx\, dy \le a_\ell \beta_\ell (A_0 + 1).$$

Since $a_\ell \to c_1 = 1/2$ and $\beta_\ell \to 1$ as $\ell \to +\infty$, the conclusion holds for φ_ℓ when ℓ is large. The proof is complete. □

C Appendix: Pointwise Convergence of $\Lambda_\delta(u)$ When $u \in W^{1,p}(\Omega)$

In this section, we prove the following result

Proposition C1 *Let $d \ge 1$, Ω be a smooth bounded open subset of $\mathbb{R}^d$, and $\varphi \in \mathcal{A}$. We have*

$$\lim_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u| \textit{ for } u \in \bigcup_{p>1} W^{1,p}(\Omega)$$

Proof We already know by Proposition 1 that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) \ge \int_\Omega |\nabla u| \quad \forall\, u \in W^{1,1}(\Omega). \tag{C1}$$

Assume now that $u \in W^{1,p}(\Omega)$ for some $p > 1$. We are going to prove that

$$\limsup_{\delta \to 0} \Lambda_\delta(u) \le \int_\Omega |\nabla u|. \tag{C2}$$

Consider an extension of u to $\mathbb{R}^d$ which belongs to $W^{1,p}(\mathbb{R}^d)$. For simplicity, we still denote the extension by u.

Clearly

$$\Lambda_\delta(u) \le \int_\Omega dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{d+1}}\, dy,$$

and thus it suffices to establish that

$$\lim_{\delta \to 0} \int_\Omega dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{d+1}}\, dy = \int_\Omega |\nabla u|\, dx. \tag{C3}$$

Using polar coordinates and a change of variables, we have, as in (2.10),

$$\begin{aligned} &\int_\Omega dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{d+1}}\, dy \\ &= \int_\Omega dx \int_0^\infty dh \int_{\mathbb{S}^{d-1}} \frac{1}{h^2} \varphi\Big(|u(x + \delta h \sigma) - u(x)|/\delta\Big)\, d\sigma. \end{aligned} \tag{C4}$$

As in (2.12), we also obtain

$$\begin{aligned} &\lim_{\delta \to 0} \frac{1}{h^2} \varphi\Big(|u(x + \delta h \sigma) - u(x)|/\delta\Big) = \frac{1}{h^2} \varphi\Big(|\nabla u(x) \cdot \sigma| h\Big) \\ &\quad \text{for a.e. } (x, h, \sigma) \in \Omega \times (0, +\infty) \times \mathbb{S}^{d-1}. \end{aligned} \tag{C5}$$

As in (2.15), we have

$$\int_\Omega dx \int_0^\infty dh \int_{\mathbb{S}^{d-1}} \frac{1}{h^2} \varphi\Big(|\nabla u(x) \cdot \sigma| h\Big)\, d\sigma = \int_\Omega |\nabla u|\, dx. \tag{C6}$$

On the other hand, since φ is non-decreasing, it follows that, for $\delta > 0$,

$$\begin{aligned} &\frac{1}{h^2} \varphi\Big(|u(x + \delta h \sigma) - u(x)|/\delta\Big) \le \frac{1}{h^2} \varphi\Big(M_\sigma(\nabla u)(x) h\Big) \\ &\qquad \text{for a.e. } (x, h, \sigma) \in \mathbb{R}^d \times (0, +\infty) \times \mathbb{S}^{d-1}, \end{aligned} \tag{C7}$$

where

$$M_\sigma(\nabla u)(x) := \sup_{\tau > 0} \int_0^1 |\nabla u(x + s\tau\sigma) \cdot \sigma|\, ds \quad \text{for } x \in \mathbb{R}^d. \tag{C8}$$

Indeed, we have

$$|u(x+\delta h\sigma)-u(x)|/\delta \le \int_0^1 h|\nabla u(x+s\delta h\sigma)\cdot\sigma|\,ds \le h\sup_{\tau>0}\int_0^1 h|\nabla u(x+s\tau\sigma)\cdot\sigma|\,ds.$$

We claim that

$$\frac{1}{h^2}\varphi\big(M_\sigma(\nabla u)(x)h\big)\,dx \in L^1\big(\Omega\times(0,+\infty)\times\mathbb{S}^{d-1}\big). \tag{C9}$$

Assuming (C9), we may then apply the dominated convergence theorem using (C4), (C5), (C6), (C7), and (C9), and conclude that (C3) holds.

To show (C9), it suffices to prove that, for all $\sigma\in\mathbb{S}^{d-1}$,

$$\int_\Omega dx\int_0^\infty \frac{1}{h^2}\varphi\big(M_\sigma(\nabla u)(x)h\big)\,dh \le C\Big(\int_{\mathbb{R}^d}|\nabla u|^p\Big)^{1/p}. \tag{C10}$$

Here and in what follows C denotes a positive constant independent of u and δ; it depends only on Ω and p. For simplicity of notation, we assume that $\sigma=e_d:=(0,\cdots,0,1)$. By a change of variables, we have

$$\begin{aligned}\int_\Omega dx\int_0^\infty \frac{1}{h^2}\varphi\big(M_{e_d}(\nabla u)(x)h\big)\,dh &= \int_\Omega \big|M_{e_d}(\nabla u)(x)\big|\,dx\int_0^\infty \varphi(t)t^{-2}\,dt\\ &=\gamma_d^{-1}\int_\Omega\big|M_{e_d}(\nabla u)(x)\big|\,dx\\ &\le C\Big(\int_\Omega\big|M_{e_d}(\nabla u)(x)\big|^p\,dx\Big)^{1/p}.\end{aligned} \tag{C11}$$

Note that

$$M_{e_d}(\nabla u)(x)=\sup_{\tau>0}\int_0^1|\partial_{x_d}u(x',x_d+s\tau)|\,ds=\sup_{\tau>0}\fint_{x_d}^{x_d+\tau}|\partial_{x_d}u(x',s)|\,ds.$$

We have

$$\begin{aligned}\int_\Omega\big|M_{e_d}(\nabla u)(x)\big|^p\,dx &\le \int_{\mathbb{R}^d}\big|M_{e_d}(\nabla u)(x)|^p\,dx\\ &=\int_{\mathbb{R}^{d-1}}dx'\int_{\mathbb{R}}\big|M_{e_d}(\nabla u)(x',x_d)\big|^p\,dx_d.\end{aligned} \tag{C12}$$

Since, by the theory of maximal functions in one dimension,

$$\int_{\mathbb{R}}\big|M_{e_d}(\nabla u)(x',x_d)\big|^p\,dx_d \le C\int_{\mathbb{R}}|\partial_{x_d}u(x',x_d)|^p\,dx_d,$$

it follows from (C12) that

$$\int_{\Omega} \left|M_{e_d}(\nabla u)(x)\right|^p \, dx \le C \int_{\mathbb{R}^d} |\nabla u(x)|^p \, dx. \tag{C13}$$

Combining (C11) and (C13) implies (C10) for $\sigma = e_d$. The proof is complete. □

Remark 10 The above proof shows that

$$\Lambda_\delta(u) \le C_p \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W^{1,p}(\Omega).$$

The idea of using the theory of maximal functions to derive a similar estimate (in a slightly different context but still for $\varphi = c_1\tilde{\varphi}_1$) is originally due to A. Ponce and J. Van Schaftingen [51]; see also H.-M. Nguyen [42].

References

1. Allaire, G.: Numerical Analysis and Optimization: An Introduction to Mathematical Modelling and Numerical Simulation. Oxford University Press, Oxford (2007)
2. Ambrosio, L., Bourgain, J., Brezis, H., Figalli, A.: Perimeter of sets and BMO-type norms. C. R. Acad. Sc. Paris **352**, 697–698 (2014)
3. Ambrosio, L., Bourgain, J., Brezis, H., Figalli, A.: BMO-type norms related to the perimeter of sets. Commun. Pure Appl. Math. **69**, 1062–1086 (2016)
4. Ambrosio, L., De Philippis, G., Martinazzi, L.: Gamma-convergence of nonlocal perimeter functionals. Manuscr. Math. **134**, 377–403 (2011)
5. Antonucci, C., Gobbino, M., Migliorini, M., Picenni, N.: Optimal constants for a non-local approximation of Sobolev norms and total variation. Preprint (2017). arXiv:1708.01231
6. Aubert, G., Kornprobst, P.: Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems? SIAM J. Numer. Anal. **47**, 844–860 (2009)
7. Bourgain, J., Brezis, H., Mironescu, P.: Another look at Sobolev spaces. In: Menaldi, J.L., Rofman, E., Sulem, A. (eds.) Optimal Control and Partial Differential Equations, A Volume in Honour of A. Bensoussan's 60th Birthday, pp. 439–455. IOS Press, Amsterdam (2001)
8. Bourgain, J., Brezis, H., Mironescu, P.: Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications. J. Anal. Math. **87**, 77–101 (2002)
9. Bourgain, J., Brezis, H., Mironescu, P.: A new function space and applications. J. Eur. Math. Soc. **17**, 2083–2101 (2015)
10. Bourgain, J., Nguyen, H.-M.: A new characterization of Sobolev spaces. C. R. Acad. Sci. Paris **343**, 75–80 (2006)
11. Braides, A.: Γ-Convergence for Beginners, Oxford Lecture Series in Mathematics and Its Applications, vol. 22. Oxford University Press, Oxford (2002)
12. Brezis, H.: How to recognize constant functions. Connections with Sobolev spaces, Volume in honor of M. Vishik, Uspekhi Mat. Nauk **57** (2002), 59–74; English translation in Russian Math. Surveys **57** (2002), 693–708
13. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext. Springer, Berlin (2010)
14. Brezis, H.: New approximations of the total variation and filters in Imaging. Atti. Accad. Naz. Lincei Rend. Lincei Mat. Appl. **26**, 223–240 (2015)
15. Brezis, H., Mironescu, P.: Sobolev Maps with Values into the Circle. Birkhäuser (**in preparation**)
16. Brezis, H., Nguyen, H.-M.: On a new class of functions related to VMO. C. R. Acad. Sci. Paris **349**, 157–160 (2011)
17. Brezis, H., Nguyen, H.-M.: Two subtle convex nonlocal approximation of the BV-norm. Nonlinear Anal. **137**, 222–245 (2016)
18. Brezis, H., Nguyen, H.-M.: The BBM formula revisited. Atti. Accad. Naz. Lincei Rend. Lincei Mat. Appl. **27**, 515–533 (2016)

19. Brezis, H., Nguyen, H.-M.: Non-convex, non-local functionals converging to the total variation. C. R. Acad. Sci. Paris **355**, 24–27 (2017)
20. Brezis, H., Nguyen, H.-M.: Paper in preparation
21. Buades, A., Coll, B., Morel, J.M.: A review of image denoising algorithms, with a new one. Multiscale Model. Simul. **4**, 490–530 (2005); Updated version in SIAM Review **52** (2010), 113–147
22. Buades, A., Coll, B., Morel, J.M.: Neighborhood filters and PDE's. Numer. Math. **105**, 1–34 (2006)
23. Buades, A., Coll, B., Morel, J.M.: Non-local means denoising. Image Process. Online **1**, 208–212 (2011)
24. Bucur, C., Valdinoci, E.: Nonlocal Diffusion and Applications, Lecture Notes of the Unione Matematica Italiana, vol. 20. Springer, Berlin (2016)
25. Caffarelli, L., Roquejoffre, J.M., Savin, O.: Nonlocal minimal surfaces. Commun. Pure Appl. Math. **63**, 1111–1144 (2010)
26. Caffarelli, L., Valdinoci, E.: Uniform estimates and limiting arguments for nonlocal minimal surfaces. Calc. Var. PDE **41**, 203–240 (2011)
27. Cai, J.-F., Dong, B., Osher, S., Shen, Z.: Image restoration, total variation, wavelet frames, and beyond. J. Am. Math. Soc. **25**, 1033–1089 (2012)
28. Chambolle, A., Lions, P.-L.: Image recovery via total variation minimization and related problems. Numer. Math. **76**, 167–188 (1997)
29. Chan, T., Esedoglu, S., Park, F., Yip, A.: Recent developments in total variation image restoration. In: Paragios, N., Chen, Y., Faugeras, O. (eds.) Mathematical Models in Computer Vision, pp. 17–30. Springer, Berlin (2005)
30. Dal Maso, G.: An Introduction to Γ-Convergence, Progress in Nonlinear Differential Equations and their Applications, vol. 8. Birkhäuser Boston Inc., Boston (1993)
31. Davila, J.: On an open question about functions of bounded variation. Calc. Var. Partial Differ. Equ. **15**, 519–527 (2002)
32. Figalli, A., Fusco, N., Maggi, F., Millot, V., Morini, M.: Isoperimetry and stability properties of balls with respect to nonlocal energies. Commun. Math. Phys. **336**, 441–507 (2015)
33. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics. CRC Press, Boca Raton (1992)
34. Fusco, N.: The quantitative isoperimetric inequality and related topics. Bull. Math. Sci. **5**, 517–607 (2015)
35. Gilboa, G., Osher, S.: Nonlocal linear image regularization and supervised segmentation. Multiscale Model. Simul. **6**, 595–630 (2007)
36. Gilboa, G., Osher, S.: Nonlocal operators with applications to image processing. Multiscale Model. Simul. **7**, 1005–1028 (2008)
37. Haddad, A., Meyer, Y.: An improvement of Rudin–Osher–Fatemi model. Appl. Comput. Harmon. Anal. **22**, 319–334 (2007)
38. Kindermann, S., Osher, S., Jones, P.W.: Deblurring and denoising of images by nonlocal functionals. Multiscale Model. Simul. **4**, 1091–1115 (2005)
39. Lee, J.S.: Digital image smoothing and the sigma filter. Comput. Vis. Graph. Image Process. **24**, 255–269 (1983)
40. Leoni, G., Spector, D.: Characterization of Sobolev and BV spaces. J. Funct. Anal. **261**, 2926–2958 (2011)
41. Leoni, G., Spector, D.: Corrigendum to "Characterization of Sobolev and BV spaces". J. Funct. Anal. **266**, 1106–1114 (2014)
42. Nguyen, H.-M.: Some new characterizations of Sobolev spaces. J. Funct. Anal. **237**, 689–720 (2006)
43. Nguyen, H.-M.: Γ-convergence and Sobolev norms. C. R. Acad. Sci. Paris **345**, 679–684 (2007)
44. Nguyen, H.-M.: Further characterizations of Sobolev spaces. J. Eur. Math. Soc. **10**, 191–229 (2008)
45. Nguyen, H.-M.: Γ-convergence, Sobolev norms, and BV functions. Duke Math. J. **157**, 495–533 (2011)
46. Nguyen, H.-M.: Some inequalities related to Sobolev norms. Calc. Var. Partial Differ. Equ. **41**, 483–509 (2011)
47. Nguyen, H.-M.: Estimates for the topological degree and related topics. J. Fixed Point Theory **15**, 185–215 (2014)
48. Paris, S., Kornprobst, P., Tumblin, J., Durand, F.: Bilateral filtering: theory and applications. Found. Trends Comput. Graph. Vis. **4**, 1–73 (2008)
49. Ponce, A.: A new approach to Sobolev spaces and connections to Γ-convergence. Calc. Var. Partial Differ. Equ. **19**, 229–255 (2004)

50. Ponce, A.: Personal communication to the authors (2005)
51. Ponce, A., Van Schaftingen, J.: Personal communication to the authors (2005)
52. Rudin, L.I., Osher, S., Fatemi, E.: Nonlinear total variation based noise removal algorithms. Phys. D **60**, 259–268 (1992)
53. Smith, S.M., Brady, J.M.: SUSAN—a new approach to low level image processing. Int. J. Comput. Vis. **23**, 45–78 (1997)
54. Van Schaftingen, J., Willem, M.: Set Transformations, Symmetrizations and Isoperimetric Inequalities, in Nonlinear Analysis and Applications to Physical Sciences, pp. 135–152. Springer Italia, Milan (2004)
55. Yaroslavsky, L.P.: Digital Picture Processing. An Introduction. Springer, Berlin (1985)
56. Yaroslavsky, L.P., Eden, M.: Fundamentals of Digital Optics. Springer, Berlin (1996)